

# Measurement of relevant elastic and damping material properties in sandwich thick-plates

Marc Rébillat

*LMS (CNRS, École Polytechnique), 91120 Palaiseau, France*

*LIMSI (CNRS), 91400 Orsay, France*

Xavier Boutillon

*LMS (CNRS, École Polytechnique), 91120 Palaiseau, France*

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## Abstract

An easy-to-implement method to measure relevant elastic and damping properties of the constituents of a sandwich structure, possibly with a heterogeneous core, is proposed. The method makes use of a one-point dynamical measurement on a thick plate. The hysteretic model for each (possibly orthotropic) constituent is written generically as “ $E(1 + j\eta)$ ” for all mechanical parameters. The estimation method of the parameters relies on a mixed experimental / numerical procedure. The frequencies and dampings of the natural modes of the plate are obtained from experimental impulse responses by means of a high-resolution modal analysis technique. This allows for considerably more experimental data to be used. Numerical modes (frequencies, dampings, and modal shapes) are computed by means of an extended Rayleigh-Ritz procedure under the “light damping” hypothesis, for given values of the mechanical parameters. Minimising the differences between the modal characteristics yields an estimation of the values of the mechanical parameters describing the hysteretic behaviour. A sensitivity analysis assess the reliability of the method for each parameter. Validations of the method are proposed by (a) applying it to virtual plates on which a finite-element model replaces the experimental modal analysis, (b) some comparisons with results obtained by static mechanical measurements, and (c) by comparing the results on different plates made of the same sandwich material.

*Key words:* Elasticity parameters estimation, Loss factor estimation, Mixed numerical / experimental procedure, High-resolution modal analysis, Thick-plate vibrations

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## 1. Introduction

For plates having a mechanical function, sandwich structures, with possibly a heterogeneous core, are often preferred to a homogeneous constitutive material because they can be made lighter. However, the relevant mechanical properties of the sandwich as a whole or even of its individual components may be difficult to predict accurately, particularly when heterogeneous cores are used or if damping is considered. Here “*relevant*” refers to the parameters that matter in the plate dynamics (see Sec. 2.1). In this paper, a method for estimating the complex moduli of elasticity of the constituents of sandwich structures having a heterogeneous core based on one dynamical test on a plate is proposed. The proposed mixed experimental / numerical procedure (for an introduction to such procedures, see for example [1, 2]) is based on the thick-plate model and frequency-independent mechanical properties of each constituent of the sandwich. It is intended to be easier to implement or to yield more parameters than other methods.

Using structural vibrations for the estimation of homogeneous material parameters is a widespread technique. Compared to static measurement campaigns which must be performed on a number of isolated samples of the sandwich components, dynamic tests leave the sandwich structure untouched and can be performed on one single panel of more or less arbitrary dimensions.

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*Email address:* rebillat@lms.polytechnique.fr (Marc Rébillat)

The problem of the estimation of solely *elasticity* parameters of a homogeneous material using plate vibrations has been widely addressed in a *thin-plate* context [1–7] and in a *thick-plate* context [8–21]. In a *thin-plate* context, only *in-plane* parameters can be estimated. Using a *thick-plate* yields some of the so-called *out-of-plane* parameters.

The problem of estimating *elasticity and damping* parameters of homogeneous materials by using point measurements [22–25] has retained some attention in a *thin-plate* context. In a *thick-plate* context, methods involving *full-field* measurements are currently available [26–28], but they are very time-consuming or need sophisticated equipment. Recently, some efforts have however been done to overcome these limitations by using point measurements instead of full-field measurements [29]. However, the experimental data used by these authors severely limits the possibilities of estimation, as discussed below.

For heterogeneous materials, such as sandwich materials, the literature addresses mainly the estimation of *elasticity* parameters, using either beams [30–32] or plates [33, 34]. A few studies have been devoted in the last years to the problem of estimating their damping properties in beams [31, 32], in plates by means of sophisticated full-field measurements [35, 36], and quite recently in plates by single point measurements [37, 38]. Since none of these studies present a sensitivity analysis, the validity of the model they used for damping is difficult to assess. The following reasoning explains how we access to significantly more experimental data than previous studies. The sensitivity analysis (Sec. 7) shows that these are determinant for some estimated parameters.

Exploiting the vibrations of sandwich panels with heterogeneous cores requires that the panel dimensions meet several conditions. In order to consider the sandwich core as homogeneous in the in-plane directions up to a given frequency  $f$ , the corresponding wavelength  $\lambda$  must contain at least 50 cells [39]. For a typical cell side-length  $s_{\text{cell}}$  and height  $h$ , this implies that the panel’s dimensions are such that  $l_{x,y} > \lambda > 50 s_{\text{cell}}$ . Moreover *out-of-plane* elasticity and damping parameters can be estimated only on thick-enough plates. In brief, the panel must be large enough and the observed dynamics must include high-enough modes, within the limit of a plate model. Due to the intrinsic dissipation of materials, high-order modes may be impossible to characterise with methods based on the Fourier-transform (FT): those are limited to modal overlaps of  $\mu \simeq 30\%$  in most implementations [35] (see Sec. 4.3 for the definition of  $\mu$ ). The high-resolution modal analysis (HRMA) technique recently developed by Ege *et al.* [40] is a successful substitute to the FT-based modal analysis techniques up to significantly higher modal overlap values.

From a methodological point of view, the present work is an extension to heterogeneous and thick plates of the work done by De Visscher *et al.* [23] on homogeneous thin plates. The proposed method is performed on large sandwich panels which meet the above conditions. Up to  $\simeq 40$  modes are extracted by means of the HRMA. This is considerably more than the 6 or 12 modes used in [29] for the determination of 6 elastic parameters. It will be shown in Sec. 7 that high-frequency modes are indeed necessary for a reliable estimation of some of these parameters. As far as damping parameters are considered, experimental data used in [29] do not allow their reliable determination (as stated by the authors and confirmed by the sensitivity analysis in Sec. 7).

The mixed experimental / numerical procedure yielding the *in-plane* and most *out-of-plane* elasticity and damping parameters (complex elasticity moduli) of the constituents of the sandwich is schematically presented in Fig. 1. The analytical model of the sandwich panel is presented in Sec. 2. Based on this model, the numerical modal frequencies  $f_n^{\text{Num}}$  and dampings  $\alpha_n^{\text{Num}}$  are derived by means of an extended Rayleigh-Ritz procedure (Sec. 3). The experimental protocol and the extraction method that yield the experimental modal frequencies  $f_n^{\text{XP}}$  and dampings  $\alpha_n^{\text{XP}}$  are presented in Sec. 4. Given the numerical and experimental data, the optimisation procedure that estimates elasticity and damping parameters of the constituents of the sandwich material is detailed in Sec. 5. The procedure is validated in Sec. 6 by means of a finite-element analysis. Measurements performed on 3 real plates are shown in Sec. 7 which provides additional validation insight as well.

## 2. A mechanical model of sandwich panels

In order to access the modal dampings and frequencies of a sandwich panel, an adapted mechanical model is needed. In this section, such a complex structure is modelled as an equivalent thick-plate under the Reissner-Mindlin hypothesis. A frequency-independent model for the materials composing the sandwich is also described. Notations used in this section are summarized by Tabs. 7, 8 and 9 in the D.

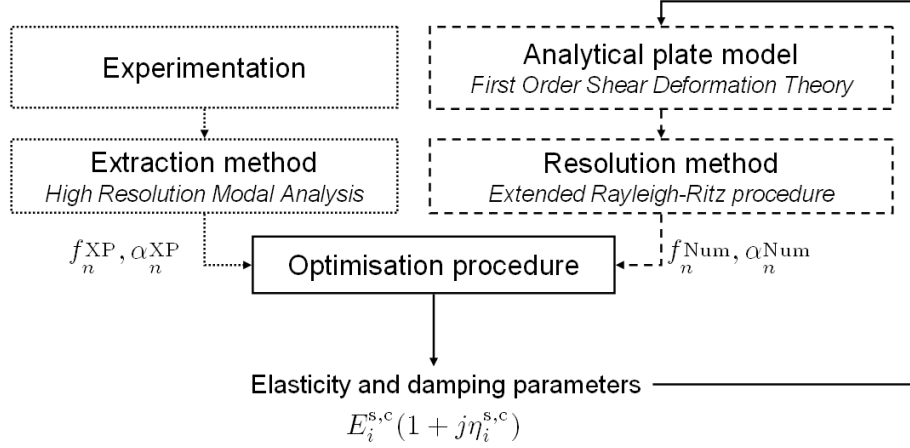


Figure 1: Overview of the proposed mixed experimental / numerical procedure: modal characteristics derived experimentally and numerically are compared; their differences tend to zero when the correct values of the mechanical parameters are reached.

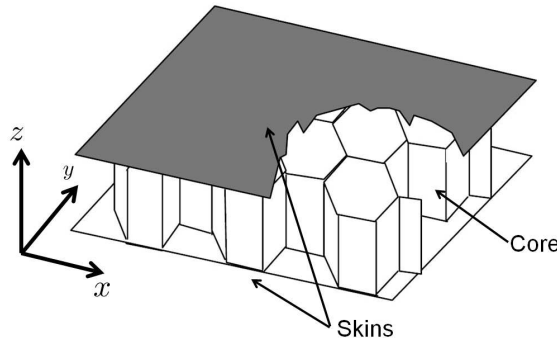


Figure 2: Geometry of the sandwich plate.

### 2.1. Hypothesis

The sandwich panel consists in two identical skins and a core (Fig. 2). The thicknesses of the core, skins, and panel are  $h^c$ ,  $h^s$ , and  $h = h^c + 2h^s$  respectively. In the following, “panel” designates the physical structure whereas “plate” refers to the idealised structure made out of the equivalent homogeneous material. The following hypotheses are made on the panel and plate:

- Displacements are small so that the materials and structures behave linearly.
- Only flexural waves of frequencies far from the delamination frequency (the frequency of the first transverse mode of the panel) are considered.
- The plate is considered to follow the Reissner-Mindlin approximations (thick-plate: first order shear deformation theory, FSDT), with no direct strain in the  $z$ -direction. As pointed out by references [41–43], the accuracy of the thick-plate model to describe the dynamics of sandwich plates having a soft core depends mainly on the thickness to length ratio and on the skins Young modulus to core shear modulus ratio:  $\frac{E_{x,y}^s}{G_{yz,xz}^c}$ . Generally, the thicker and the softer the core, the less the thick-plate model is appropriate for modelling purposes as compared to higher-order models. In [42, Fig. 4], a comparison between thick-plate theory (FSDT) and high-order shear

deformation theory (HSDT) is performed on the first modal frequency of a sandwich panel with a soft core. At least for orders of magnitude considered, it could be extended to a higher modal frequency of wavelength  $\lambda$  by considering the ratio  $\frac{2h}{\lambda}$  instead of  $\frac{h}{\lambda}$  (the “ $h$ ” written here is defined in Fig. 2 and corresponds to the writing “ $2h$ ” in [42]). According to [42, Fig. 4(c) and 4(d)], the difference between FSTD and HSDT does not exceed 5 % for  $\frac{E_{x,y}^s}{G_{yz,xz}^c} \simeq 100$  and a  $\frac{2h}{\lambda}$  ratio less than 0.08. The plates that are considered in this paper exhibit a  $\frac{E_{x,y}^s}{G_{yz,xz}^c} \simeq 100$  ratio and the highest modes under consideration are characterized by  $\frac{2h}{\lambda} \simeq 0.08$ . A thick-plate model based on FSTD is thus appropriate enough in the present study.

- The wavelengths  $\lambda$  include at least 50 cells. According to Burton *et al.* [39], this ensures that errors on the modal frequencies of the plate (with a homogeneous equivalent core) are less than 2% when compared to those of the panel as computed by various finite-elements models.

The skin and core materials are considered as homogeneous, orthotropic in the  $x$ - and  $y$ - directions.

The formalism chosen for describing the hysteretic behaviour is that of complex moduli  $\underline{E} = E(1 + j\eta)$  which do not depend on the frequency (see the model of materials in section 2.2). The Young’s and shear moduli and the Poisson coefficient of the core are  $\underline{E}_x^c$ ,  $\underline{E}_y^c$ ,  $\underline{E}_z^c$ ,  $\underline{G}_{xy}^c$ ,  $\underline{G}_{xz}^c$ ,  $\underline{G}_{yz}^c$ ,  $\underline{\nu}_{yx}^c$ ,  $\underline{\nu}_{xz}^c$ ,  $\underline{\nu}_{yz}^c$  and  $\underline{\nu}_{xy}^c$ . The same parameters for the skins are denoted by the index “s”. The properties of the homogeneous material equivalent to the whole sandwich are denoted by the index “H”. Additional symmetry relationships are given in section 2.2.

The following hypotheses are made on the sandwich panel:

- The sandwich panel is symmetric with respect to its mid-plane.
- Skins are thin compared to the core and the core is softer than the skins so that shear stress in the skins can be ignored:  $\frac{h^s}{G_{xz}^s} \ll \frac{h^c}{G_{xz}^c}$  (and the same in the  $y$  direction).
- The core is very soft ( $E_x^c \ll E_x^s$ ,  $E_y^c \ll E_y^s$  and  $G_{xy}^c \ll G_{xy}^s$ ). Given the generic expression of the moduli of the homogeneous equivalent material  $E^H = \left(\frac{h^c}{h}\right)^3 E^c + \left[1 - \left(\frac{h^c}{h}\right)^3\right] E^s$ , this ensures that all in-plane stress in the plate are entirely due to those in the skins.

According to these hypotheses, there is no stress associated with  $\underline{E}_z^{c,s,H}$ ,  $\underline{\nu}_{xz}^{c,s,H}$ ,  $\underline{\nu}_{yz}^{c,s,H}$ ,  $\underline{G}_{xz}^s$ ,  $\underline{G}_{yz}^s$ ,  $\underline{E}_x^c$ ,  $\underline{E}_y^c$ ,  $\underline{G}_{xy}^c$ ,  $\underline{\nu}_{xy}^c$ ,  $\underline{\nu}_{yx}^c$  which are ignored in what follows. The relevant remaining mechanical parameters describing such a plate are thus [36]:

- *In-plane* parameters (bending of the skins) :  $\underline{E}_x^s$ ,  $\underline{E}_y^s$ ,  $\underline{G}_{xy}^s$ ,  $\underline{\nu}_{yx}^s$ ,  $\underline{\nu}_{xy}^s$ .
- *Out-of-plane* parameters (shearing in the core) :  $\underline{G}_{xz}^c$ ,  $\underline{G}_{yz}^c$  (excluding  $\underline{E}_z^c$ ).

In the rest of the article, it will thus be understood that “*elastic parameters*” means “*relevant elastic parameters for the dynamics of thick-plates*”.

These hypotheses are generally fulfilled in common sandwich panels. Typical orders of magnitude for parameters are:

$$\begin{cases} h^s/h^c \simeq 10^{-1} \\ E_x^c/E_x^s \simeq E_y^c/E_y^s \simeq G_{xy}^c/G_{xy}^s \simeq 10^{-4} \end{cases} \quad (1)$$

## 2.2. Model of the materials

The damping of plate vibrations has different origins. In the present study, it is assumed that panels vibrate below their coincidence acoustical frequencies [44]. Consequently, damping due to acoustical radiation in surrounding air is very small compared to the structural damping [45]. Among the different structural damping models, the standard hysteretic model (which is frequency-independent, see for example [46, 47]) has been retained. The relationship

between the stress  $\epsilon^\gamma$  and the strain  $\sigma^\gamma$  in each  $\gamma$ -material ( $\gamma = s, c, \text{ or } H$ ) involves 7 complex numbers and can be written, to first order in  $\eta^\gamma$  as:

$$\sigma^\gamma = \begin{bmatrix} E_x^\gamma(1 + j\eta_x^\gamma) & \nu_{yx}^\gamma E_x^\gamma[1 + j(\eta_{\nu_{yx}}^\gamma + \eta_x^\gamma)] & 0 & 0 & 0 \\ \nu_{xy}^\gamma E_y^\gamma[1 + j(\eta_{\nu_{xy}}^\gamma + \eta_y^\gamma)] & E_y^\gamma(1 + j\eta_y^\gamma) & 0 & 0 & 0 \\ 0 & 0 & G_{xz}^\gamma(1 + j\eta_{xz}^\gamma) & 0 & 0 \\ 0 & 0 & 0 & G_{yz}^\gamma(1 + j\eta_{yz}^\gamma) & 0 \\ 0 & 0 & 0 & 0 & G_{xy}^\gamma(1 + j\eta_{xy}^\gamma) \end{bmatrix} \epsilon^\gamma \quad (2)$$

The symmetry of the strain/stress relation adds the following relationships  $\nu_{xy}^\gamma E_y^\gamma = \nu_{yx}^\gamma E_x^\gamma$  and  $\eta_y^\gamma = \eta_{\nu_{xy}}^\gamma + \eta_y^\gamma = \eta_{\nu_{yx}}^\gamma + \eta_x^\gamma$  which leaves 12 independent real parameters to be identified for each material (24 altogether).

### 2.3. Equivalent thick-plate

Under the hypothesis and for the orders of magnitude given in section 2.1, the sandwich panel behaves in the low frequency range like a homogeneous thick-plate [48]. The thickness of the plate is chosen to be  $h$ . Its mechanical properties are given in Eq. (3) and (4) as functions of the mechanical and geometrical properties of the skins and the core.

$$\begin{cases} E_x^H = E_x^s \left[ 1 - \left( \frac{h^c}{h} \right)^3 \right] & E_y^H = E_y^s \left[ 1 - \left( \frac{h^c}{h} \right)^3 \right] & \nu_{xy}^H = \nu_{xy}^s \\ G_{xy}^H = G_{xy}^s \left[ 1 - \left( \frac{h^c}{h} \right)^3 \right] & G_{xz}^H = G_{xz}^c & G_{yz}^H = G_{yz}^c \end{cases} \quad (3)$$

$$\begin{cases} \eta_x^H = \eta_x^c \frac{E_x^c}{E_x^s} \left( \frac{h^c}{h} \right)^3 + \eta_x^s \left[ 1 - \left( \frac{h^c}{h} \right)^3 \right] & \eta_y^H = \eta_y^c \frac{E_y^c}{E_y^s} \left( \frac{h^c}{h} \right)^3 + \eta_y^s \left[ 1 - \left( \frac{h^c}{h} \right)^3 \right] \\ \eta_{xy}^H = \eta_{xy}^c \frac{G_{xy}^c}{G_{xy}^s} \left( \frac{h^c}{h} \right)^3 + \eta_{xy}^s \left[ 1 - \left( \frac{h^c}{h} \right)^3 \right] & \eta_{xz}^H = \eta_{xz}^c & \eta_{yz}^H = \eta_{yz}^c & \eta_v^H = \eta_v^s \end{cases} \quad (4)$$

The 12 independent real parameters  $\{E_x^H, \eta_x^H, E_y^H, \eta_y^H, G_{xy}^H, \eta_{xy}^H, G_{xz}^H, \eta_{xz}^H, G_{yz}^H, \eta_{yz}^H, \nu_{xy}^H, \eta_v^H\}$  are to be estimated. Their knowledge yields the elastic and damping properties of each layer of the sandwich panel provided that the 12-equation system formed by Eqs. (3) and (4) is invertible. A sufficient condition is:

$$\eta_x^c \frac{E_x^c}{E_x^s} \ll \eta_x^s \quad \eta_y^c \frac{E_y^c}{E_y^s} \ll \eta_y^s \quad \eta_{xy}^c \frac{G_{xy}^c}{G_{xy}^s} \ll \eta_{xy}^s \quad (5)$$

since  $\frac{E_x^c}{E_x^s} \ll 1$ ,  $\frac{E_y^c}{E_y^s} \ll 1$ , and  $\frac{G_{xy}^c}{G_{xy}^s} \ll 1$  (see section 2.1). This condition is not satisfied only if the  $\eta^c$ -coefficients are several orders of magnitude larger than the  $\eta^s$ -ones. This is not the case here and rarely the case in general<sup>1</sup>. Consequently, the estimation of  $E_x^H$ , etc ... yields an estimation of the mechanical properties of the skin and core materials.

### 2.4. Potential, kinetic and dissipated energies in the equivalent thick-plate

Within the frame of the first order Reissner-Mindlin theory [49, Chap. 3], the displacements  $\{u, v, w\}$  in the  $\{x, y, z\}$ -directions respectively can be written within a good approximation (see below) as:

$$u(x, y, z) = -z\Phi_x(x, y) \quad v(x, y, z) = -z\Phi_y(x, y) \quad w(x, y, z) = w_0(x, y) \quad (6)$$

The potential energy of the plate is:

<sup>1</sup> It can be the case when skins are made of metal and the core is made of paper honeycombs or of viscoelastic foam.

$$\begin{aligned}
U &= \frac{1}{2} \iiint_V (\boldsymbol{\sigma}^H)^T \boldsymbol{\epsilon}^H d\tau \\
&= \frac{1}{2} \iint_S \left[ D_1 \left( \frac{\partial \Phi_x}{\partial x} \right)^2 + D_2 \left( \frac{\partial \Phi_x}{\partial x} \frac{\partial \Phi_y}{\partial y} \right) + D_3 \left( \frac{\partial \Phi_y}{\partial y} \right)^2 + D_4 \left( \Phi_y^2 - 2\Phi_y \frac{\partial w_0}{\partial y} + \left( \frac{\partial w_0}{\partial y} \right)^2 \right) + \dots \right. \\
&\quad \left. D_5 \left( \Phi_x^2 - 2\Phi_x \frac{\partial w_0}{\partial x} + \left( \frac{\partial w_0}{\partial x} \right)^2 \right) + D_6 \left( \left( \frac{\partial \Phi_x}{\partial y} \right)^2 + 2 \frac{\partial \Phi_x}{\partial y} \frac{\partial \Phi_y}{\partial x} + \left( \frac{\partial \Phi_y}{\partial x} \right)^2 \right) \right] dx dy
\end{aligned} \tag{7}$$

with

$$\begin{aligned}
D_1 &= \frac{E_x^H h^3}{12(1 - \nu_{xy} \nu_{yx})} & D_2 &= \frac{\nu_{xy} E_y^H h^3}{6(1 - \nu_{xy} \nu_{yx})} & D_3 &= \frac{E_y^H h^3}{12(1 - \nu_{xy} \nu_{yx})} \\
D_4 &= 2\kappa_{yz}^2 h G_{yz}^H & D_5 &= 2\kappa_{xz}^2 h G_{xz}^H & D_6 &= \frac{G_{xy}^H h^3}{6}
\end{aligned} \tag{8}$$

The shear correction factors  $\kappa_{yz}^2$  and  $\kappa_{xz}^2$  account for the fact that Eq. (6) is an approximation: the (functional) angles  $\Phi_x$  and  $\Phi_y$  depend lightly on  $z$  and sections of the plate do not remain plane in the flexural deformation. The values  $\kappa_{yz} = \kappa_{xz} = 1$  have been chosen according to the recommendations of [50] for sandwich panels.

By definition, the fraction of energy lost during one cycle  $\mathcal{T}$  is:

$$\Delta U = - \int_{\mathcal{T}} \left[ \iiint_V (\boldsymbol{\sigma}^H)^T \frac{\partial \boldsymbol{\epsilon}^H}{\partial t} d\tau \right] dt \tag{9}$$

Based on section 2.2,  $\Delta U$  can then be expressed as:

$$\begin{aligned}
\Delta U &= -\pi \iint_S \left[ \eta_x^H D_1 \left( \frac{\partial \Phi_x}{\partial x} \right)^2 + \eta_y^H D_2 \left( \frac{\partial \Phi_x}{\partial x} \frac{\partial \Phi_y}{\partial y} \right) + \eta_y^H D_3 \left( \frac{\partial \Phi_y}{\partial y} \right)^2 + \eta_{yz}^H D_4 \left( \Phi_y^2 - 2\Phi_y \frac{\partial w_0}{\partial y} + \left( \frac{\partial w_0}{\partial y} \right)^2 \right) + \dots \right. \\
&\quad \left. \eta_{xz}^H D_5 \left( \Phi_x^2 - 2\Phi_x \frac{\partial w_0}{\partial x} + \left( \frac{\partial w_0}{\partial x} \right)^2 \right) + \eta_{xy}^H D_6 \left( \left( \frac{\partial \Phi_x}{\partial y} \right)^2 + 2 \frac{\partial \Phi_x}{\partial y} \frac{\partial \Phi_y}{\partial x} + \left( \frac{\partial \Phi_y}{\partial x} \right)^2 \right) \right] dx dy
\end{aligned} \tag{10}$$

The kinetic energy  $T$  of the system is given in Eq. (11) as a function of  $\Phi_x$ ,  $\Phi_y$ , and  $w_0$ . In this expression,  $\rho^H$  is the density of the equivalent homogeneous thick plate. It is given by  $h\rho^H = h^c\rho^c + 2h^s\rho^s$ .

$$T = \frac{\rho^H \omega^2}{2} \iiint_{(V)} [u^2 + v^2 + w^2] d\tau = \frac{\rho^H \omega^2}{2} \iint_{(S)} \left[ \frac{h^3}{12} (\Phi_x^2 + \Phi_y^2) + h w_0^2 \right] dx dy \tag{11}$$

### 3. Estimation of modal parameters by an extended Rayleigh-Ritz procedure

In order to compare experimental results to numerical simulations, it is necessary to evaluate the frequencies  $f_n^{\text{Num}}$  and damping factors  $\alpha_n^{\text{Num}}$  of the numerical modes corresponding to the sandwich panel. The dynamics of the plate is given by the hypotheses listed in section 2.1, by Eqs. (2), and by the boundary conditions. Under the ‘‘light damping’’ hypothesis, which assumes that modal shapes and frequencies are unchanged by the addition of damping, the frequencies  $f_n^{\text{Num}}$  of the numerical modes are easily accessible. The problem consists thus in evaluating the relationships between the  $\alpha_n^{\text{Num}}$  damping factors and the  $\eta^H$  loss-factors. Notations used in this section are summarized by Tab. 10 in the D.

#### 3.1. Light damping hypothesis

The honeycomb sandwich panel is considered here as a non-conservative system  $\mathcal{P}_{\text{NC}}$  having  $N$  degrees of freedom. The associated conservative system, without hysteretic damping and having also  $N$  degrees of freedom, is denoted  $\mathcal{P}_{\text{C}}$ . The  $N$  modes of  $\mathcal{P}_{\text{C}}$  have their modal shapes denoted by  $\xi_n^{\text{C}}$  and their real modal frequencies denoted by  $f_n^{\text{C}}$ . The  $N$  modes of  $\mathcal{P}_{\text{NC}}$  have their modal shapes denoted by  $\xi_n^{\text{NC}}$  and their complex modal frequencies denoted by  $f_n^{\text{NC}}$ .

If  $\mathcal{P}_{\text{NC}}$  is lightly damped, it can be shown [51] that  $\xi_n^{\text{NC}} \simeq \xi_n^{\text{C}}$  and that  $f_n^{\text{NC}} \simeq f_n^{\text{C}} + j\alpha_n$  to first order. The “light damping” hypothesis thus assumes that modal shapes and real parts of the frequencies are unchanged by the addition of damping. This hypothesis has been shown to be acceptable for values of material loss factors lower than 0.1 [52]. This assumption is similar to the assumption made in the “Modal Strain Energy” approach used to model sandwich panels having visco-elastic cores [52–54].

Let  $U_n^{\text{NC}}$  be the potential energy associated with the  $n^{\text{th}}$  mode of  $\mathcal{P}_{\text{NC}}$  for a maximum vibrational amplitude of 1 on the plate. It varies in time as  $\exp(-2\alpha_n t)$  so that the energy lost by this mode during one cycle,  $\Delta U_n^{\text{NC}}$ , is:

$$\Delta U_n^{\text{NC}} = -2 \frac{\alpha_n}{f_n^{\text{C}}} U_n^{\text{NC}} \quad (12)$$

Since  $\mathcal{P}_{\text{C}}$  and  $\mathcal{P}_{\text{NC}}$  have the same modal shapes, *i.e.*  $\xi_n^{\text{NC}} \simeq \xi_n^{\text{C}}$ , and the potential energy depends only on the modal shapes (see Eq. (7)), then  $U_n^{\text{NC}} = U_n^{\text{C}}$ . And since for the conservative system  $\mathcal{P}_{\text{NC}}$ , the equality  $U_n^{\text{C}} = T_n^{\text{C}}$  is true, one then obtains:

$$\Delta U_n^{\text{NC}} = -2 \frac{\alpha_n}{f_n^{\text{C}}} T_n^{\text{C}} \quad (13)$$

Thanks to the light damping hypothesis, Eq. (13) gives a straightforward way to obtain the modal dampings  $\alpha_n$  from  $\Delta U_n^{\text{NC}}$ ,  $T_n^{\text{C}}$  and,  $f_n^{\text{C}}$ .

### 3.2. Derivation of $f_n^{\text{Num}}$

A Rayleigh-Ritz procedure has been used to derive numerically the modal frequencies  $f_n^{\text{Num}}$  and the mode shapes  $\xi_n^{\text{Num}}$  of  $\mathcal{P}_{\text{C}}$ . To this end, the generalised-displacement fields  $\Phi_x(x, y)$ ,  $\Phi_y(x, y)$ , and  $w_0(x, y)$  are projected on the elements of an orthonormal polynomial basis of order  $Q$  satisfying partially the free-free boundary conditions [55, 56]:

$$\Phi_x(x, y) = \sum_{i,j} L_{ij} p_i(x) p_j(y) \quad \Phi_y(x, y) = \sum_{i,j} M_{ij} p_i(x) p_j(y) \quad w_0(x, y) = \sum_{i,j} N_{ij} p_i(x) p_j(y) \quad (14)$$

where the polynomials  $p_i(\kappa)$  are generated as described in A.

This procedure generates a new set of  $3Q^2$  generalised displacements  $L_{ij}$ ,  $M_{ij}$  and  $N_{ij}$ . The next step consists in writing the kinetic and potential energies  $T$  and  $U$  which have been expressed as functions of  $\Phi_x$ ,  $\Phi_y$ , and  $w_0$  in section 2.4. The Hamilton principle reads as:

$$\forall (i, j) \in [0, Q-1]^2 : \frac{\partial(T-U)}{\partial L_{ij}} = 0 \quad \frac{\partial(T-U)}{\partial M_{ij}} = 0 \quad \frac{\partial(T-U)}{\partial N_{ij}} = 0 \quad (15)$$

The above system of  $3Q^2$  linear equations can be re-written as  $[\mathbf{K} - 4\pi^2 f^2 \mathbf{M}] \mathbf{q} = 0$ . The expressions of the partial derivatives of  $U$  with respect to  $L_{ij}$ ,  $M_{ij}$ ,  $N_{ij}$  yield  $\mathbf{K}$  while the partial derivatives of  $T$  with respect to  $L_{ij}$ ,  $M_{ij}$ ,  $N_{ij}$  yield  $\mathbf{M}$ . The explicit expressions for these partial derivatives as functions of the  $p_i$ -s and of the generalised displacements are given in B. The resolution of this eigenvalue problem gives a straightforward access to the modal frequencies  $f_n^{\text{Num}}$  and mode shapes  $\xi_n^{\text{Num}}$  of  $\mathcal{P}_{\text{C}}$ . Under the light damping hypothesis,  $\xi_n^{\text{Num}}$  and  $f_n^{\text{Num}}$  found for  $\mathcal{P}_{\text{C}}$  are also the modal shapes and frequencies of  $\mathcal{P}_{\text{NC}}$  (see Sec.3.1).

### 3.3. Derivation of $\alpha_n^{\text{Num}}$

Introducing the modal coefficients  $\xi_n^{\text{Num}}$  (expressed in the  $\{L_{ij}, M_{ij}, N_{ij}\}$  system of coordinates) into Eqs. 14 yields analytical expressions for the  $\Phi_x$ ,  $\Phi_y$ , and  $w_0$  modal fields and also for their  $x$ - and  $y$ -derivatives. For each of the  $N$  modes, the potential, lost, and kinetic energies can be written by introducing these expressions into Eqs. 7, 10, and 11:

$$\forall n \in [1, N] : \quad T_n^{\text{C}} = 4\pi^2 (f_n^{\text{Num}})^2 t_n \quad U_n^{\text{NC}} = U_n^{\text{C}} = \sum_{k=1}^6 D_k u_k^n \quad \Delta U_n^{\text{NC}} = -\pi \sum_{k=1}^6 \eta_k D_k u_k^n \quad (16)$$

where the subscripts  $\{1, 2, 3, 4, 5, 6\}$  of  $\eta$  stand for  $\{x, v, y, yz, xz, xy\}$  respectively. The coefficients  $t_n$  and  $u_k^n$  are given explicitly in C. They depend on the geometry and mass parameters of the plate, and are quadratic in modal shapes



$\xi_n^{\text{Num}}$ . Note that  $t_n$  can be considered as half the modal mass for some normalised displacement. The product  $D_k u_k^n$  represents the  $k$ -contribution to the  $n$ -th modal stiffness (where  $k$  stands for  $x$ ,  $y$ ,  $yz$ ,  $xz$ , or  $xy$ ).

The expression (17) of the modal dampings  $\alpha_n^{\text{Num}}$  can be deduced from the Eqs. (13) and the last two expressions of (16) or, equivalently, by (12) and the first two expressions of (16):

$$\alpha_n^{\text{Num}} = -\frac{f_n^{\text{Num}} \Delta U_n^{\text{NC}}}{2T_n^{\text{C}}} = \frac{1}{8\pi f_n^{\text{Num}} t_n} \sum_{k=1}^6 \eta_k D_k u_k^n \quad \text{or} \quad \alpha_n^{\text{Num}} = -\frac{f_n^{\text{Num}} \Delta U_n^{\text{NC}}}{2U_n^{\text{C}}} = \frac{f_n^{\text{Num}} \pi}{2 \sum_{m=1}^6 D_m u_m^n} \sum_{k=1}^6 \eta_k D_k u_k^n \quad (17)$$

One can notice that  $\alpha_n^{\text{Num}}$  is a linear combination of the  $\eta_k$ . This set of equations is a generalisation to all the modes of a thick-plate of the expression given by De Visscher *et al.* [23, Eq. (13)] for three particular modes of a thin plate. This expression is also a generalization to all the loss factors of the expression established by Johnson *et al.* [53] for sandwich structures having a visco-elastic core.

#### 4. Estimation of modal parameters by high-resolution modal analysis

In this section, the experimental protocol that has been followed to obtain impulse responses and to extract the modal frequencies  $f_n^{\text{XP}}$  and the damping factors  $\alpha_n^{\text{XP}}$  is presented. This protocol combines the procedures and implementation details presented by Rébillat *et al.* [57, 58] and Ege *et al.* [40].

##### 4.1. Experimental setup

Throughout all measurements, panels were suspended by thin wires in order to ensure free-free boundary conditions. Light panels were acoustically excited by an electro-dynamical loudspeaker driven by a wide-band electrical signal [27]. The velocity response was measured in one corner of each panel with a laser Doppler vibrometer (Ometron VH300+ type 8329). Eventual non-linearities arising from the loudspeaker were removed and the impulse response of the panel was reconstructed [57, 58]. Since heavy panels can hardly be excited by acoustical means, impact excitations were also used; in this case, the acceleration resulting from the impact was recorded with a light accelerometer fixed in the vicinity of one corner of the panel. The impulse response was obtained after deconvolution with the nearly-impulsive force signal [40]. Since no nodal line goes through corners of a free vibrating plate, all excited modes contribute to the resulting impulse response.

##### 4.2. High resolution modal analysis

It is hypothesised that the experimental data are corrupted by additive noise. Thus, an impulse response  $h(t)$  is mathematically represented as a sum of decaying exponentials (natural modes) and measurement noise  $b(t)$  as in Eq. (18). Each contribution of a natural mode is characterised by its amplitude  $A_n^{\text{XP}}$ , frequency  $f_n^{\text{XP}}$ , damping  $\alpha_n^{\text{XP}}$  and phase  $\phi_n^{\text{XP}}$ .

$$\begin{aligned} h(t) &= \sum_{n=1}^N A_n^{\text{XP}} \exp(-\alpha_n^{\text{XP}} t) \cos(2\pi f_n^{\text{XP}} t + \phi_n^{\text{XP}}) + b(t) \\ &= \sum_{n=1}^N \frac{A_n^{\text{XP}}}{2} \left[ \exp(-\alpha_n^{\text{XP}} t + j2\pi f_n^{\text{XP}} t + j\phi_n^{\text{XP}}) + \exp(-\alpha_n^{\text{XP}} t - j2\pi f_n^{\text{XP}} t - j\phi_n^{\text{XP}}) \right] + b(t) \end{aligned} \quad (18)$$

In order to extract experimental modal frequencies  $f_n^{\text{XP}}$  and damping  $\alpha_n^{\text{XP}}$  from  $h(t)$ , the recently developed ‘‘High Resolution Modal Analysis’’ (HRMA) [40] has been applied and is briefly sketched below. The signal is projected onto two subspaces: the subspace spanned by the sinusoids (*signal subspace*) and its supplementary (*noise subspace*) according to the ESPRIT (Estimation of Signal Parameters via Rotational Invariance Techniques) algorithm [59]. The frequencies  $f_n^{\text{XP}}$  and dampings  $\alpha_n^{\text{XP}}$  of a given number of modes are the eigenvalues of a matrix obtained after some computation on the observed signal. The amplitudes  $A_n^{\text{XP}}$  and phases  $\phi_n^{\text{XP}}$  are estimated afterwards by a least-mean-square method.



In the ESPRIT procedure, the dimensions of both subspaces must be chosen *a priori* and the quality of the estimation significantly relies on a proper choice for these parameters. The best choice for the dimension of the signal subspace is the number of exponentials (twice the number of decaying sinusoids, or real modes, see Eq. (18)). This number can be estimated, before the analysis, by means of the ESTER technique [60].

To improve the performance of the ESPRIT algorithm, signals are split into several frequency-bands [61], thus reducing the number of modes to be processed. In order to limit computation time, the responses of the band-pass filters are frequency-shifted and down-sampled.

A block-diagram describing the different steps involved in HRMA is shown in Fig. 3.

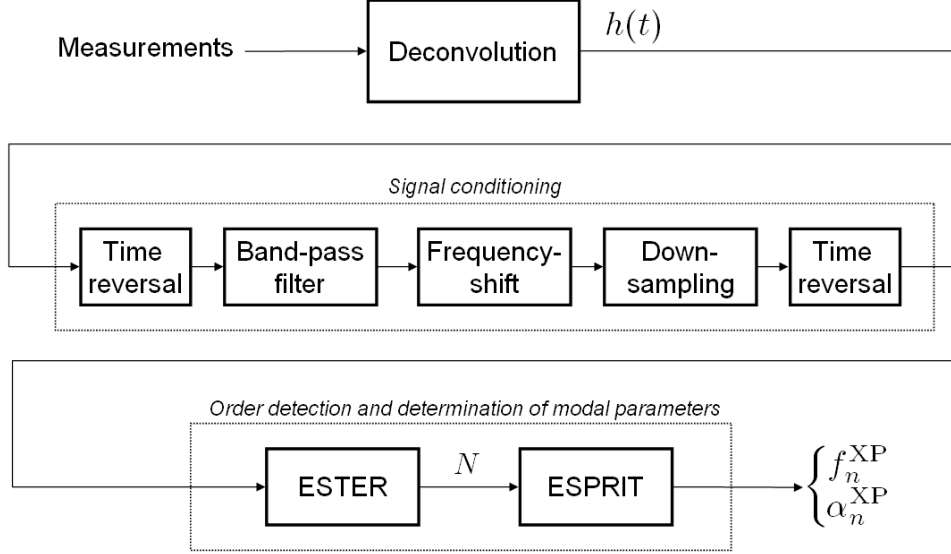


Figure 3: Block diagram of the high resolution modal analysis method, adapted from [40]. The first block is described in [57].

#### 4.3. Uncertainties in modal parameters estimation

To give an overview of the precision offered by the HRMA, this method is applied to a synthetic signal obtained by adding two decaying exponentials of equal amplitudes to white noise. The sampling frequency is  $f_s = 44.1$  kHz. The two modal frequencies are 592 and 596 Hz, very close one from each other. For relatively important modal dampings, these two modes overlap in the frequency-domain and therefore, their frequencies and dampings are difficult to estimate with methods based on the Fourier transform (FT). The modal overlap factor  $\mu$  (*i.e.* the ratio between the half-power modal bandwidth  $\Delta f_{-3\text{ dB}}$  and the average modal spacing  $\Delta f_{\text{mode}}$ ) quantifies this phenomenon [40]. If the modal damping  $\alpha$  is the same for both modes,  $\mu$  is:

$$\mu = \frac{1}{\Delta f_{\text{mode}}} \frac{\alpha}{\pi} \quad (19)$$

In practice, the FT cannot efficiently separate modes when  $\mu > 30\%$  [35]. For HRMA, this upper limit depends on the signal/noise ratio and on the number of components which are retained in the pre-conditioning step. As an example, estimations of modal frequencies and dampings have been performed on the synthetic signal with various  $\alpha$ -values (corresponding to modal overlaps from  $\mu = 1\%$  to  $150\%$ ) and a signal-to-noise ratio (SNR) increasing from 10 dB to 50 dB. SNR is understood here as  $20 \log_{10} \left( \frac{S_{\text{RMS}}}{B_{\text{RMS}}} \right)$ , where  $S_{\text{RMS}}$  is the RMS value of signal in absence of noise and  $B_{\text{RMS}}$  the RMS value of noise in absence of signal. For each couple  $\{\mu, \text{SNR}\}$ , modal parameters were estimated 50 times. For each mode of the synthetic signal, the uncertainty in frequency or damping is defined as the

mean of the absolute values of the relative error between the original and the estimated data. These uncertainties, expressed in dB, are shown as contour plots in Fig. 4.3.

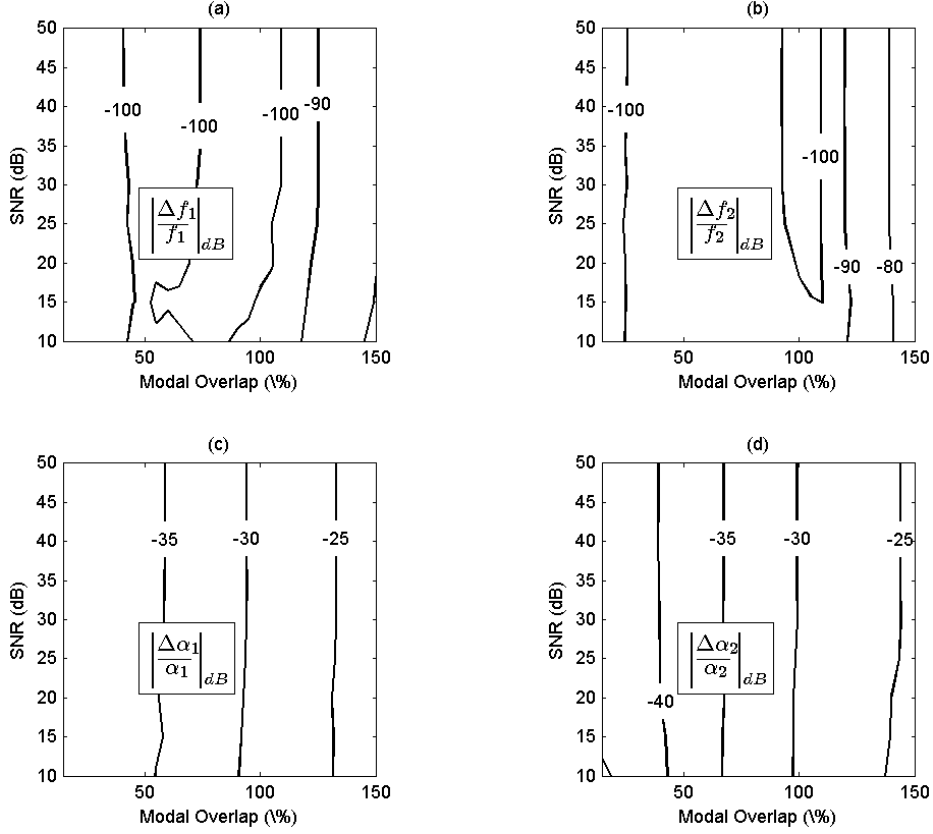


Figure 4: Contour plots of the uncertainty on the estimated “modal” frequencies and dampings of a synthetic signal containing two decaying sinusoids as a function of the modal overlap and the signal-to-noise ratio. The uncertainty is the mean of 50 absolute values of the relative error between original and estimated data, for 50 different realisations of the noise. (a): uncertainties on the frequency of the first mode. (b): uncertainties on the frequency of the second mode. (c) and (d): *idem* relatively to the dampings.

It can be seen in Fig. 4.3 that the uncertainties are very small for both dampings and frequencies even for high modal overlaps and low SNR. In general, the uncertainty increases with  $\mu$  and decreases slightly when SNR increases. The HRMA gives better estimations of the eigenfrequencies and dampings than the FFT for a wide range of  $\mu$  and SNR. Since the modal overlap  $\mu$  generally increases in the impulse response of a plate, and the SNR decreases with the frequency, the HRMA gives access to significantly more modes than the usual FT-based modal-analysis techniques and is thus of great interest in the present context.

## 5. Optimisation procedure

This section describes how to derive, in two steps, the complex moduli of elasticity of the homogenised equivalent material of the sandwich plate  $\{E_x^H, \eta_x^H, E_y^H, \eta_y^H, G_{xy}^H, \eta_{xy}^H, G_{xz}^H, \eta_{xz}^H, G_{yz}^H, \eta_{yz}^H, \nu_{xy}^H, \eta_v^H\}$  from the experimental and numerical values of the modal frequencies and dampings  $f_n^{XP}, f_n^{Num}, \alpha_n^{XP}$ , and  $\alpha_n^{Num}$ .

### 5.1. Elastic properties

The estimation of the elasticity parameters  $\{E_x^H, E_y^H, G_{xy}^H, G_{xz}^H, G_{yz}^H, \nu_{xy}^H\}$  is done by comparing the experimental and numerical modal frequencies. The estimation problem to solve is non-linear and several orders of magnitude are involved in the properties values. The following cost-function was used:

$$C_f = \sum_{n=1}^N \left( \frac{f_n^{\text{XP}} - f_n^{\text{Num}}}{f_n^{\text{XP}}} \right)^2 \quad (20)$$

A steepest-descent (with backtracking) algorithm [62] using rigidities  $\{D_1, D_2, D_3, D_4, D_5, D_6\}$  as design variables has been chosen. In the present case, the coefficients of the gradient can be easily derived analytically, making the method easy to implement and computationally light.

Estimation results obtained by gradient methods are known to be very dependent on the initial values of the parameters. To minimise the influence of the starting point, the following initialisation strategy for the rigidities has been chosen:

1. Initial values of in-plane rigidities  $D_1, D_2, D_3$  and  $D_6$  are the most influential; they were derived from the three lowest modal frequencies of the panel, as proposed in [22].
2. Initial values of out-of-plane rigidities  $D_4$  and  $D_5$  are less critical; homogenisation theory proposed by Gibson [63] for honeycomb core sandwich panels is used. This theory requires a value for the elasticity moduli of the material composing the honeycomb core. The first estimation was based on static tests.

In the following examples, 10 iterations were enough to reach convergence:  $10^{-7}$  for the gradient.

### 5.2. Damping properties

As can be seen in Eq. (17), modal dampings depend linearly on the loss factors  $\{\eta_x^H, \eta_y^H, \eta_{xy}^H, \eta_{xz}^H, \eta_{yz}^H, \eta_v^H\}$  once the rigidities have been found. The estimation of the loss factors is therefore much easier than that of the elasticity parameters. A simple least-square optimisation procedure is sufficient to estimate the loss factors from the modal dampings.

The following cost-function has been chosen:

$$C_\alpha = \sum_{n=1}^N \left( \frac{\alpha_n^{\text{XP}} - \alpha_n^{\text{Num}}}{\alpha_n^{\text{XP}}} \right)^2 \quad (21)$$

The optimisation procedure is not iterative and needs no particular initialisation.

### 5.3. Determination of the order $Q$ of the polynomial basis and of the number $N$ of included modal parameters

Two parameters have to be chosen in order to apply the optimisation procedures described in Secs. 5.1 and 5.2. These methodological parameters are the order  $Q$  of the polynomial basis associated with the extended Rayleigh-Ritz procedure (see Sec. 3.2) and the number  $N$  of modal parameters (frequencies or dampings) to be included in the optimisation procedure.

The parameters  $Q$  and  $N$  can be chosen differently for the estimation of the elastic material properties and for the estimation of the material loss factors respectively. For the estimation of the elastic material properties, the parameters  $Q_f$  and  $N_f$  that minimize the cost functions  $C_f$  given by Eq. (20) are chosen. For the estimation of the damping properties, the parameters  $Q_\alpha$  and  $N_\alpha$  that minimize the cost functions  $C_\alpha$  given by Eq. (21) are chosen. Selecting the optimal  $Q$  and  $N$  values is done empirically by running the optimisations for different values of these parameters, typically in the ranges  $N \in \{20, 40\}$  and  $Q \in \{10, 18\}$ .

### 5.4. Sensitivity analysis

For the estimation procedures described in sections 5.1 and 5.2 to be efficient, modal frequencies and dampings must convey a sufficient amount of information relative to each parameter to be estimated. In other words, modal dampings and frequencies have to be sensitive to the parameters of interest.

Relevant sensitivities can be defined and calculated analytically. The sensitivity of the modal value  $\tau$  to the parameter  $\gamma$  is noted  $S_\gamma^\tau$  and defined by Eq.(22): if  $\gamma$  is increased by 1 %,  $\tau$  increases by  $S_\gamma^\tau$  %.

$$S_\gamma^\tau = \frac{\Delta \tau}{\Delta \gamma} \frac{\gamma}{\tau} = \frac{\frac{\Delta \tau}{\tau}}{\frac{\Delta \gamma}{\gamma}} \quad (22)$$

	$l_x$ (m)	$l_y$ (m)	$h$ (m)		$\rho$ (kg/m <sup>3</sup> )	
HVP	0.4	0.6	$4 \times 10^{-3}$		700	
VSP	0.4	0.6	Core	Skin	Core	Skin
			$4 \times 10^{-3}$	$0.2 \times 10^{-3}$	40	700

Table 1: Geometry and constituent densities of the homogeneous virtual plate (HVP) and of the virtual sandwich-plate (VSP).

According to Eq. (16) and with the same notations, the sensitivity  $S_{D_k}^{f_n}$  of the  $n^{\text{th}}$  modal frequency  $f_n$  to the rigidity  $D_k \in \{D_1, D_2, D_3, D_4, D_5, D_6\}$  can be written as:

$$S_{D_k}^{f_n} = \frac{D_k u_{nk}}{8\pi^2 f_n^2 t_n} \quad (23)$$

Similarly, using Eq. (17), the sensitivity  $S_{\eta_k}^{\alpha_n}$  of the  $n^{\text{th}}$  modal damping  $\alpha_n$  to the loss factor  $\eta_k \in \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$  can be written as:

$$S_{\eta_k}^{\alpha_n} = \frac{\eta_k D_k u_{nk}}{4\pi f_n \alpha_n t_n} \quad (24)$$

The amount of information relative to one given parameter and contained in one given mode can be easily quantified with Eq. (23) and Eq. (24). Examples are given in Figs. 7, 10, and 12 (see Sec. 7).

## 6. Validation of the estimation procedure

A validation of the mechanical model and procedures given in sections 2, 3, and 5 is proposed as follows: the experimental results of the modal analysis are replaced by those of the simulation of a finite-element model (FEM) of two virtual plates with known properties. The modes of a homogeneous thick plate as modelled using the FEM are first compared to those given by the extended Rayleigh-Ritz procedure applied to the mechanical model given in section 2. The method for deriving elasticity and damping parameters as sketched in Fig. 1 is afterwards validated on a virtual sandwich plate.

### 6.1. Finite-element model of the virtual plates

To validate the extended Rayleigh-Ritz procedure applied to the mechanical model, a homogeneous thick virtual plate was designed. A FE-model of the sandwich panel has also been built to test the accuracy of the estimation method. The chosen sandwich plate is made of 3 homogeneous layers and is symmetrical with respect to its mid-plane. Geometrical, mechanical and mass parameters of the two plates are given in Tab. 1 and Tab. 2.

For the two virtual plates under study, the finite element model is built on a 2D rectangular mesh made of 60 by 60 regularly spaced points. This value is justified at the end of this section. At each point an 8-nodes shell element is placed with a linear expansion of the in-plane displacements in the thickness coordinate and a constant transverse displacement through the thickness (COQ8 of the Cast3M code [64]). Each of these elements possess 6 degrees of freedom (the translations in the x, y, and z directions and the rotations around the x-, y- and z-axes). In the case of the virtual sandwich plate, the three-layers are modelled as one equivalent layer as in Sec. 2.3.

### 6.2. Modal frequencies and dampings of the virtual plates

Finite-element modelling and the associated computations have been performed using Cast3M [64], a free software developed by the French Centre for Atomic Energy (CEA). This software is used here as it allows to find the complex modes (modal shapes  $\xi_n^{\text{FEM}}$ , dampings  $\alpha_n^{\text{FEM}}$  and frequencies  $f_n^{\text{FEM}}$ ) of a problem put in the form:  $\mathbf{K} + j\omega\mathbf{C} - \omega^2\mathbf{M} = \mathbf{0}$ . In this formulation,  $\mathbf{M}, \mathbf{C}$  and  $\mathbf{K}$  must be real matrices to be accepted by the VIBC function of the Cast3M code. Complex modes are then found by solving a complex valued generalized eigenvalue problem using a QZ-algorithm. According to the possibilities offered by Cast3M, modal frequencies and dampings of the virtual plates are computed using the following procedure:

		$\underline{E}_x$	$\underline{E}_y$	$\nu_{xy}$	$\underline{G}_{xy}$	$\underline{G}_{xz}$	$\underline{G}_{yz}$
HVP	Real part	4 GPa	5 GPa	0.33	1 GPa	$10^{-2}$ GPa	$10^{-2}$ GPa
	Loss factor (%)	2	5	1	0.5	1	1
VSP Skins	Real part	4 GPa	5 GPa	0.33	1 GPa	1 GPa	1 GPa
	Loss factor (%)	2	5	1	0.5	1	1
VSP Core	Real part	$1 \times 10^{-3}$ GPa	$1 \times 10^{-3}$ GPa	0.33	$1 \times 10^{-3}$ GPa	$1 \times 10^{-2}$ GPa	$3 \times 10^{-2}$ GPa
	Loss factor (%)	1	1	1	1	3	5

Table 2: Mechanical parameters chosen for the the homogeneous virtual plate (HVP) and for the virtual sandwich-plate (VSP)

1. The conservative system is described according to the constitutive model of section 2.2 with no hysteretic damping taken into account. A mass matrix  $\mathbf{M}$  and a real stiffness matrix  $\mathbf{K}'$  are deduced from this model.
2. The  $N$  first modal frequencies  $\{f_n^{\text{FEM}}\}_{n \in [1, N]}$  of the conservative system are computed by solving, in the Fourier domain, the real-valued eigenvalue problem  $\mathbf{K}' - \omega^2 \mathbf{M} = 0$ .
3. The non-conservative system is described according to the constitutive model of section 2.2, including hysteretic damping. A mass matrix  $\mathbf{M}$  and a complex stiffness matrix  $\mathbf{K} = \mathbf{K}' + j\mathbf{K}''$  are deduced from this model.
4. The “light damping hypothesis,” is retained. The real part of the modal frequencies of the non-conservative system are thus already known (see Sec. 3.1).
5. For each one of the  $N$  first modes of the non-conservative system, the following operations are then performed:
  - (a) The dynamic equation of the dissipative system are formulated, in the Fourier domain, as  $-\omega^2 \mathbf{M}\mathbf{q} + j\omega \mathbf{C}\mathbf{q} + \mathbf{K}'\mathbf{q} = \mathbf{0}$  with  $\mathbf{C} = \mathbf{K}''/(2\pi f_n^{\text{FEM}})$ . The problem is thus formulated as expected by the VIBC function with  $\mathbf{C}$  real but frequency-dependent. Its important to notice that this equation models correctly the hysteretic damping model described in Sec. 2.2 only near  $\omega \simeq 2\pi f_n^{\text{FEM}}$ .
  - (b) The modal loss factor  $\alpha_n$  is obtained as the imaginary part of the eigenvalue of this new problem solved near  $\omega \simeq 2\pi f_n^{\text{FEM}}$ .

For the homogeneous virtual plate, increasing the number of elements above 60 elements per side results in less than a 1 % relative variation of the 35 first modal frequencies (conservative and non conservative cases) and in less than 0.4 % of the 35 first modal dampings. The same convergence is observed for the 3-layer virtual sandwich plate. Thus, 60 elements per side are enough to ensure the desired precision on the analysis of the first 35 modes of the two plates.

### 6.3. Validation of the extended Rayleigh-Ritz resolution of the mechanical model

Comparing the modal frequencies and dampings given by the extended Rayleigh-Ritz resolution method (18-order) and by the FE-model for the homogeneous thick virtual plate provides an estimate of the reliability of the retained mechanical model coupled with the extended Rayleigh-Ritz resolution method for the first 35 modes.

The plate under study in the present section is not a sandwich panel as the one modelled in Sec.2 but a homogeneous thick plate. For such homogeneous plates, values for the shear correction factors are usually chosen between 2/3 and 5/6 instead of 1, which is recommended only for sandwich panels [50]. Values of  $\kappa_{yz} = \kappa_{xz} = 0.7$  have been arbitrarily chosen here in the range [0.666, 0.833].

With shear correction factors  $\kappa_{yz} = \kappa_{xz} = 0.7$ , the mean absolute difference between FE-results and Rayleigh-Ritz results is lower than 1.2 % for modal frequencies and lower than 25 % for modal dampings. The larger error on modal dampings may be explained as follows. The damping matrix  $\mathbf{C}$  provided to the FEM results from the writing of the stiffness matrix and is not necessarily diagonal when expressed in the basis of the conservative modes. By contrast, the extended Rayleigh-Ritz procedure accounts for dissipation by associating one damping coefficient  $\alpha_n^{\text{Num}}$  to each mode, neglecting by construction any coupling between conservative modes.

### 6.4. Estimation results for the 3-layer virtual sandwich-plate

A 3-layer virtual sandwich-plate has been used to validate the estimation procedure described in Fig. 1. Since, for this 3-layer virtual sandwich-plate, convergence problems were encountered with the steepest-descent algorithm,

a simplex search method [65] was used instead in this case (function “*fminsearch*” in Matlab<sup>TM</sup>). The initialisation procedure remains the same as the one described in Sec. 5.1. For all other optimisations, the steepest-descent algorithm is used.

Based on the first  $N = 35$  modal frequencies given by the FEM and using a Rayleigh-Ritz order  $Q = 16$ , the estimated values of the elasticity parameters are compared to the original values given to the FEM. The loss factors have been estimated with  $N = 28$  modes and a model order  $Q = 18$ . The estimated mechanical parameters are presented in Fig. 5 for each layer of the sandwich.

The residual mismatch between the results of estimation and the original values is discussed here. The mean absolute value  $\left\langle \left| \frac{\Delta f_n}{f_n} \right| \right\rangle$  of the relative difference between experimental and numerical modal frequencies is 2.6 %. For the dampings, the residual mismatch  $\left\langle \left| \frac{\Delta \alpha_n}{\alpha_n} \right| \right\rangle$  is 21.6 %. These orders of magnitude, as compared to the one obtained in Sec. 6.3, suggest that the assumption that a 3-layer sandwich plate can be modelled as a simple homogeneous thick plate is correct in the frequency range under study.

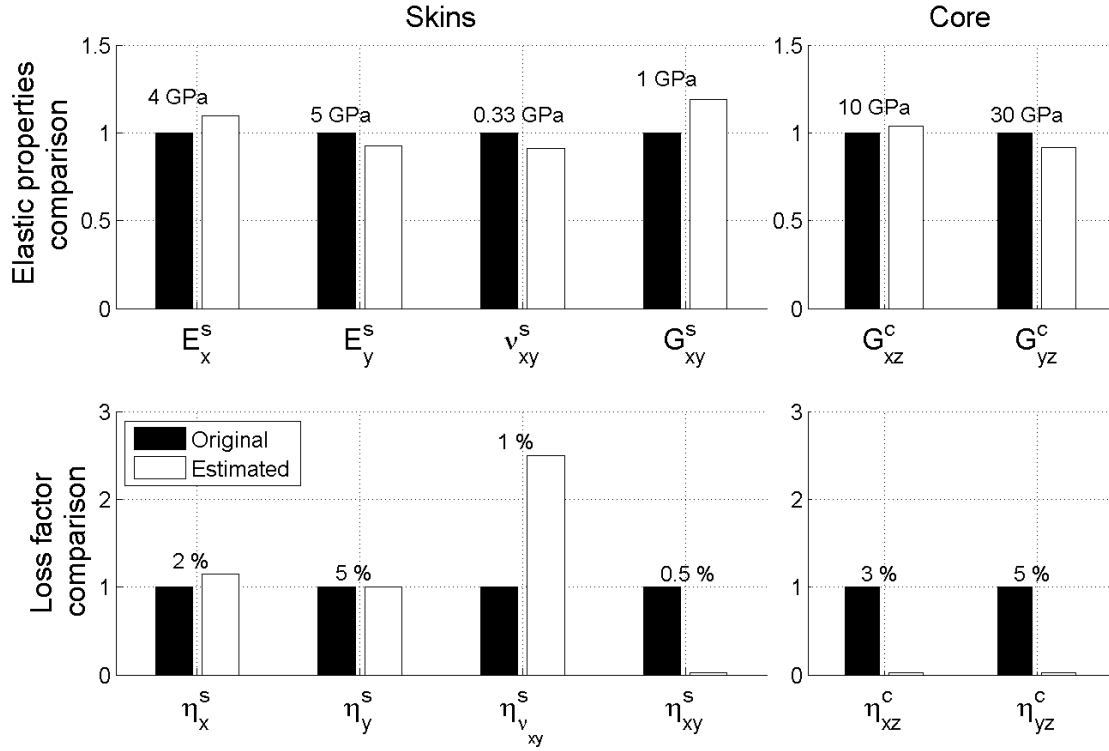


Figure 5: Comparison between the values of the mechanical parameters used in the FEM (Original), and their estimated values (Estimated) for each constituent (Skins, Core) of the virtual sandwich plate. Original parameters are represented as black bars with their numerical value indicated above. White bars represent the ratio of the estimated to the original parameters. Elasticity parameters have been estimated with 35 modes and with a model order  $Q = 16$ . Loss-factors have been estimated with 28 modes and with a model order  $Q = 18$ .

It can be seen in Fig. 5 that the agreement between estimated and original parameters is globally very good. In-plane elasticity parameters of the skins and out-of-plane elasticity parameters of the core are estimated with a mean absolute relative error of 10.2 %. Principal in-plane loss-factors  $\eta_x^s$  and  $\eta_y^s$  are estimated with a comparable accuracy of 7.5 %. The imaginary part of  $\nu_{xy}^s$  is largely overestimated while the imaginary part of  $G_{xy}^s$  is underestimated. However, the overestimation of one parameter may be the result of the underestimation of the other, by compensation. The imaginary parts of  $G_{xz}^c$  and  $G_{yz}^c$  are assigned zero values by the estimation process. The fact that zeros values are found illustrates the limitations of the thick-plate model under Reissner-Minldin hypothesis. Physically, this

underestimation is due to the fact that only a marginal part of the total energy-loss per cycle is dissipated through the mechanical couplings described by  $\underline{G}_{xz}^c$  and  $\underline{G}_{yz}^c$ . Modal dampings factors are thus here not very sensitive to these material loss factors.

A complete validation study should have established the validity limits of the estimation method. Even though this is not what has been done here, the above results suggest that the mixed numerical/experimental procedure is potentially an accurate tool for the estimation of the main elasticity moduli and loss-factors of 3-layers sandwich plates.

## 7. Experimental results

### 7.1. Plate specimens

Three different sandwich panels with heterogeneous cores have been investigated. The first two – HC<sub>1</sub> and HC<sub>2</sub> – are rectangular lightweight honeycomb sandwich panels. Their skins and honeycomb cores are made of epoxy and paper. The third panel, made of two stainless steel sheets (skins) and two bidirectionally corrugated steel layers with a 20 % relative density (core), is denoted CC (for "corrugated core"). The geometry and mass parameters of each panel are given in Tab. 3.

	$l_x$ (cm)	$l_y$ (cm)	$h^s$ (mm)	$h^c$ (mm)	$s_{\text{cell}}$ (mm)	$\rho^c$ (kg/m <sup>3</sup> )	$\rho^s$ (kg/m <sup>3</sup> )
HC <sub>1</sub>	39.15	59.10	0.20	4.88	4.0	37.8	713
HC <sub>2</sub>	80.00	99.95	0.20	3.80	4.0	37.8	713
CC	17.78	22.86	0.20	1.48	1.0	2164	7800

Table 3: Geometry and constituent-densities of three sandwich panels HC<sub>1</sub>, HC<sub>2</sub>, and CC. The characteristic side-length of the core-cells is  $s_{\text{cell}}$ .

The  $\frac{E_{xy}^s}{G_{yz,xz}^c} \simeq 100$  criterium (see Sec. 2.1) that must be satisfied turns out to be met for all the plates that have been tested. According to section 1, the sandwich core can be considered as homogeneous in the in-plane directions up to a given frequency  $f_{\text{max}}$  if the corresponding wavelength  $\lambda^{\text{min}}$  contains at least 50 cells [39]. For a typical cell side-length  $s_{\text{cell}}$  and height  $h$ , this implies that  $\frac{\lambda^{\text{min}}}{s_{\text{cell}}} > 50$ . Moreover, plates must be thick-enough in order that out-of-plane elasticity parameters and loss-factors be estimated, but not too thick for the thick-plate theory to remain valid. This implies  $\frac{2h}{\lambda^{\text{min}}} < 0.08$  [42]. The validity of these assumptions will be discussed.

### 7.2. Results for panel HC<sub>1</sub>

Panel HC<sub>1</sub> was acoustically excited [27] and 46 modes were identified. Elasticity parameters and loss-factors were estimated with  $N = 40$  modes and a model order  $Q = 14$  using the steepest-descent algorithm of Sec. 5.1. The estimated parameters of the equivalent homogeneous plate and the corresponding skin and core parameters are given in Tab. 4.

The equivalent plate corresponding to panel HC<sub>1</sub> was found to be slightly orthotropic. This is a consequence of the laminated skins and of the orthotropy of the honeycomb structure. One can also notice that very low values are found for the loss factors associated with the Poisson ratio: little energy is dissipated via the Poisson effect in panel HC<sub>1</sub>. Also, out-of-plane loss factors are relatively larger than the in-plane loss factors; this denotes that dissipation in honeycomb core structures is larger for out-of-plane shearing than for bending.

Static tensile tests have been performed on two samples of the skin material in the x- and y-directions respectively. The results are:  $E_x^s = 5.3 \pm 0.5$  GPa,  $\nu_{xy}^s = 0.28 \pm 0.04$ ,  $E_y^s = 7.3 \pm 0.7$  GPa,  $\nu_{yx}^s = 0.27 \pm 0.04$ . These values are in excellent agreement with the values estimated using the proposed method. This constitutes an additional validation for the proposed method.

The optimisation procedure consists in minimising the difference between the experimental modal frequencies and dampings and numerical modal frequencies and dampings. The residual differences are presented in Fig. 6 and provide an estimation of the reliability of the method. One can observe that there is a very low relative difference between



		$\underline{E}_x$	$\underline{E}_y$	$\underline{\nu}_{xy}$	$\underline{G}_{xy}$	$\underline{G}_{xz}$	$\underline{G}_{yz}$
Equivalent plate	Real part	1.0 GPa	1.4 GPa	0.25	0.46 GPa	12 MPa	26 MPa
	Loss factor (%)	1.5	1.3	0	1.2	5.5	4.1
Core	Real part	-	-	-	-	12 MPa	26 MPa
	Loss factor (%)	-	-	-	-	5.5	4.1
Skins	Real part	4.8 GPa	6.8 GPa	0.25	5.6 GPa	-	-
	Loss factor (%)	1.5	1.3	0	1.2	-	-
Skins	Tensile tests	5.3 GPa	7.3 GPa	0.28	-	-	-
		$\pm 0.5$ GPa	$\pm 0.7$ GPa	$\pm 0.04$			

Table 4: Estimated mechanical parameters for panel HC<sub>1</sub>. Parameters relative to the skins and the core are obtained after inversion of Eqs. (3) and (4).

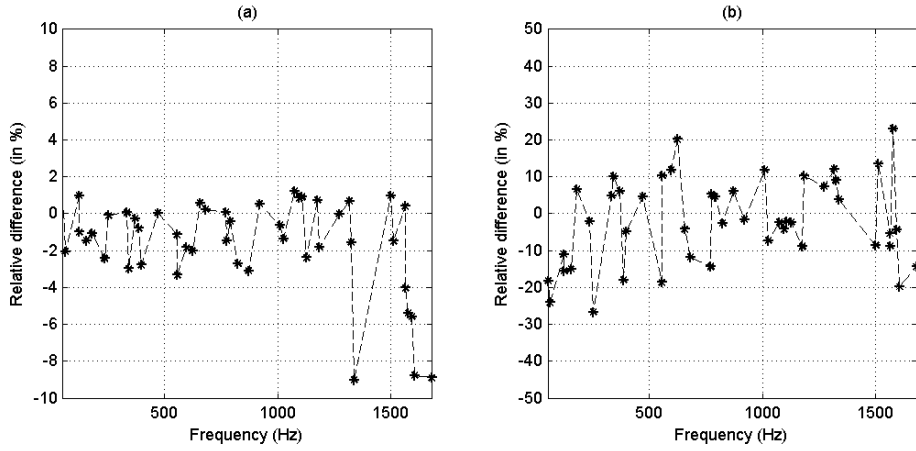


Figure 6: Residual differences on eigenfrequencies (a) and dampings (b) for panel HC<sub>1</sub>. Numerical values have been obtained using elasticity and damping parameters from Tab. 4.

the measured and numerical modal frequencies:  $\left| \frac{\Delta f_n}{f_n} \right| = 2\%$ . Thus, the homogeneous thick plate model based on the Reissner-Mindlin hypothesis agrees with the real dynamical behavior of panel HC<sub>1</sub>. Moreover, there is also a low relative difference between measured and numerical modal dampings:  $\left| \frac{\Delta \alpha_n}{\alpha_n} \right| = 10\%$ . The frequency-independent loss factors combined with the “light damping” hypothesis appears to be a good model for the constitutive material of panel HC<sub>1</sub>.

Based on the estimated values given in Tab. 4, the modal shapes can be computed by means of the extended Rayleigh-Ritz procedure. In the  $y$ -direction, panel HC<sub>1</sub> has a maximum of 8 nodal lines in the frequency range under consideration. This corresponds to  $4.5\lambda_{\min}^y$ : the shortest wavelength is 13 cm and contains  $\lambda_{\min}^y/s_{\text{cell}} \approx 30$  cells. In the  $x$ -direction, there are up to 6 nodal lines:  $\lambda_{\min}^x = 11$  cm,  $\lambda_{\min}^x/s_{\text{cell}} \approx 28$  cells. Theoretically, this is hardly sufficient for the core to be considered as homogeneous. The condition  $\frac{2h}{\lambda_{\min}} \leq 0.08$  is met for almost all modes since  $\frac{2h}{\lambda_{\min}^y} = 0.08$  in the  $y$ -direction and  $\frac{2h}{\lambda_{\min}^x} = 0.1$  in the  $x$ -direction. However, a very good agreement is observed in Fig. 6 between the homogeneous model and the experimental values. Therefore, 30 cells per wavelength may be enough in the present case for the core to be considered as homogeneous. The thick-plate theory also seems sufficient for  $\frac{2h}{\lambda_{\min}} = 0.1$ .

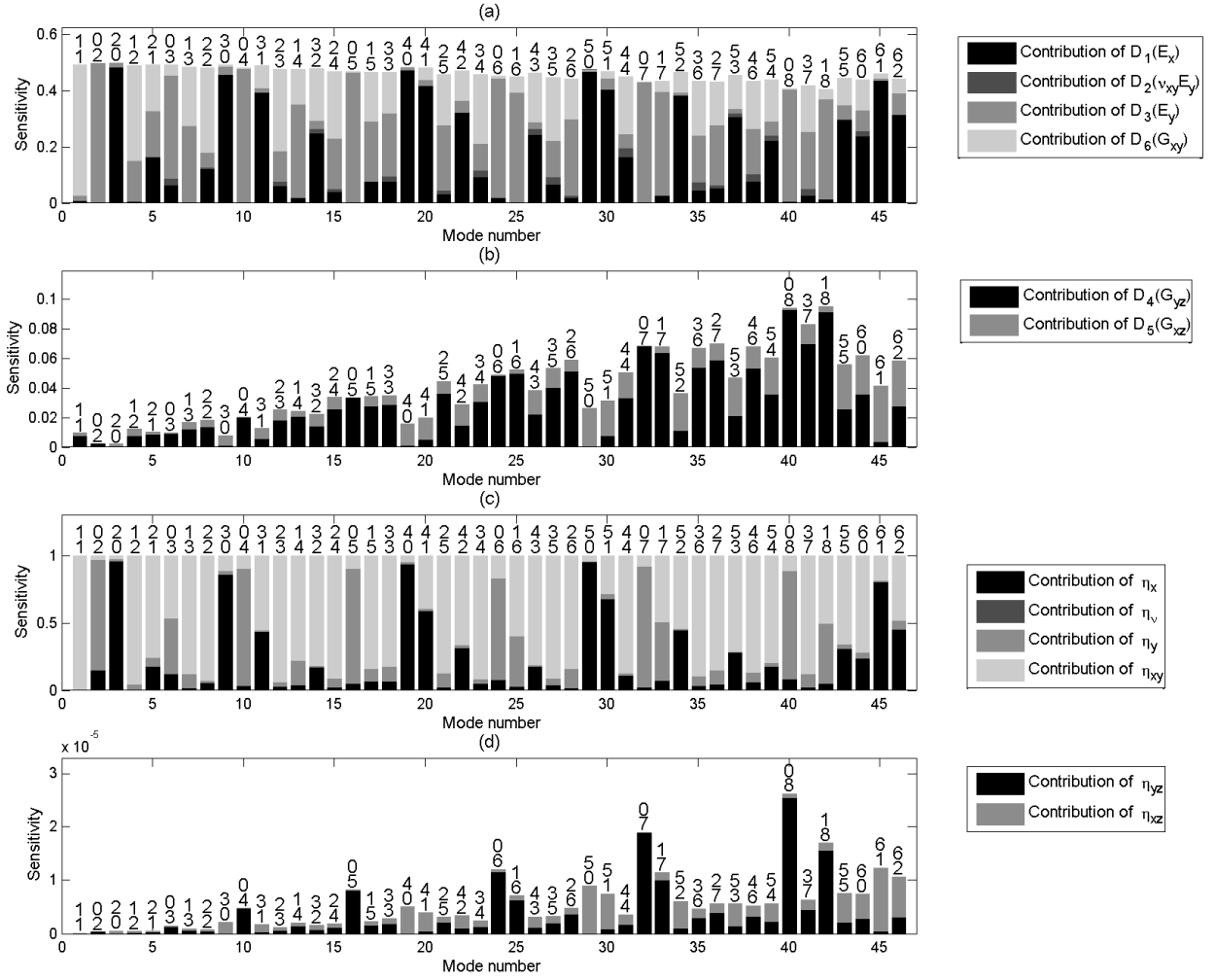


Figure 7: Sensitivities of the modal frequencies to the in-plane (a) and the out-of-plane (b) elasticity parameters. Sensitivities of the modal dampings to the in-plane (c) and the out-of-plane (d) loss factors for panel HC<sub>1</sub>. Modes are identified by the numbers of their nodal lines in the x- and y-directions (top and bottom numbers respectively, on top of each bar).

Sensitivities of the modal frequencies and dampings to the in-plane and out-of-plane mechanical parameters for panel HC<sub>1</sub> are shown in Fig. 7. Modal frequencies and dampings are sensitive to all the in-plane elasticity and damping parameters. The estimated in-plane mechanical properties are thus reliable. Sensitivities to the out-of plane mechanical properties are relatively important. This ensure a high degree of confidence for the estimated values of  $\text{Re}(\bar{G}_{xz}^H)$  and  $\text{Re}(\bar{G}_{yz}^H)$  since a large number of sensitive modes are involved in the optimisation procedure. Sensitivity to the out-of-plane loss factors is one order of magnitude lower. Thus, estimated out-of-plane loss factors are less reliable than in-plane loss factors.

### 7.3. Results for panel HC<sub>2</sub>

Panel HC<sub>2</sub> was excited by an impact hammer and 26 modes were extracted. Elasticity parameters and loss-factors were estimated with  $N = 26$  modes and a model order  $Q = 14$  using the steepest-descent algorithm of Sec. 5.1. The estimated parameters of the equivalent homogeneous plate are given in Tab. 5.

As can be seen in Fig. 8, in-plane and out-of plane elasticity and damping parameters are similar to those of panel HC<sub>1</sub>. Theoretically, if the cores of the two plates were made of the same material (which is not known with

		$\underline{E}_x$	$\underline{E}_y$	$\underline{\nu}_{xy}$	$\underline{G}_{xy}$	$\underline{G}_{xz}$	$\underline{G}_{yz}$
Equivalent plate	Real part	1.0 GPa	1.2 GPa	0.27	0.48 GPa	13 MPa	28 MPa
	Loss factor (%)	1.0	1.1	0.0	3.2	44	30
Skins	Real part	3.8 GPa	4.7 GPa	0.27	1.9 GPa	-	-
	Loss factor (%)	1.0	1.1	0.0	1.2	-	-
Core	Real part	-	-	-	-	13 MPa	28 MPa
	Loss factor (%)	-	-	-	-	44	30

Table 5: Estimated mechanical parameters for panel HC<sub>2</sub>. Parameters relative the skins and the core are obtained after inversion of Eqs. (3) and (4)

certainty but seems to be the case),  $\text{Re}(\underline{G}_{xz}^H)$  and  $\text{Re}(\underline{G}_{yz}^H)$  should be equal for both panels, according to Eq. (3). This is verified here with a good degree of precision. This robustness against the size of test-panels constitutes an additional indication that the proposed method is reliable with regard to material properties.

The residual differences  $\left\langle \left| \frac{\Delta f_n}{f_n} \right| \right\rangle$  and  $\left\langle \left| \frac{\Delta \alpha_n}{\alpha_n} \right| \right\rangle$  for panel HC<sub>2</sub> are shown in Fig. 9. They are approximately 3.8 % and 16.7 % respectively. Plate and material models can be considered as appropriate for these honeycomb core sandwich panels (panels HC<sub>1</sub> and HC<sub>2</sub>).

Based on the estimated values given in Tab. 5, the modal shapes of the extracted modes can be computed with the extended Rayleigh-Ritz procedure (Sec. 3.2). In the  $y$ -direction, panel HC<sub>2</sub> has a maximum of 5 nodal lines in the frequency range under consideration: its side-length is  $3\lambda_y^{\min}$  and  $\lambda_y^{\min} = 33$  cm, containing  $\simeq 80$  cells. In the  $x$ -direction, panel HC<sub>2</sub> has a maximum of 5 nodal lines:  $\lambda_x^{\min} = 26.5$  cm,  $\frac{\lambda_x^{\min}}{s_{\text{cell}}} \simeq 65$  cells. This is more than sufficient

for the core to be considered as homogeneous. Moreover in both directions,  $\frac{2h}{\lambda_{\min}} \simeq 2.4 \times 10^{-2} < 0.08$ , which is theoretically sufficient for modal frequencies to be predicted using thick-plate theory. The core of panel HC<sub>2</sub> can be considered as homogeneous and all the modal frequencies can be predicted using thick-plate theory.

Sensitivities of the modal frequencies and dampings to the in-plane and out-of-plane mechanical parameters for panel HC<sub>2</sub> are shown in Fig. 10. Sensitivities to the elastic out-of-plane mechanical properties are lower for panel HC<sub>2</sub> than for panel HC<sub>1</sub>. However, since results obtained on panel HC<sub>2</sub> are in close agreement with those obtained with panel HC<sub>1</sub>, this suggests that a sensitivity of  $\simeq 10^{-2}$  may still yield reliable results. Very low sensitivities to the out-of-plane loss factors explain that values of loss factors estimated on panel HC<sub>2</sub> deviate significantly from the values obtained with panel HC<sub>1</sub>.

#### 7.4. Results for panel CC

Panel CC was excited with an impact hammer. Elasticity material parameters have been estimated with  $N = 35$  modes and a model order  $Q = 14$  using the steepest-descent algorithm of Sec. 5.1. Loss factors have been estimated with  $N = 23$  modes and a model order  $Q = 13$ . The estimated parameters of the equivalent homogeneous plate are given in Tab. 6.

		$\underline{E}_x$	$\underline{E}_y$	$\underline{\nu}_{xy}$	$\underline{G}_{xy}$	$\underline{G}_{xz}$	$\underline{G}_{yz}$
Equivalent plate	Real part	117 GPa	102 GPa	0.33	43 GPa	77 GPa	163 GPa
	Loss factor (%)	0.1	0.1	0.0	0.1	0.0	0.7
Skins	Real part	229 GPa	200 GPa	0.33	84 GPa	-	-
	Loss factor (%)	0.1	0.1	0.0	0.1	-	-
Core	Real part	-	-	-	-	77 GPa	163 GPa
	Loss factor (%)	-	-	-	-	0.0	0.7

Table 6: Estimated mechanical parameters for panel CC. Parameters relative the skins and the core are obtained after inversion of Eqs. (3) and (4)

At first, it can be seen from Tab. 6 that the real parts of  $\underline{E}_x$  and  $\underline{E}_y$  for skins match standard values for the

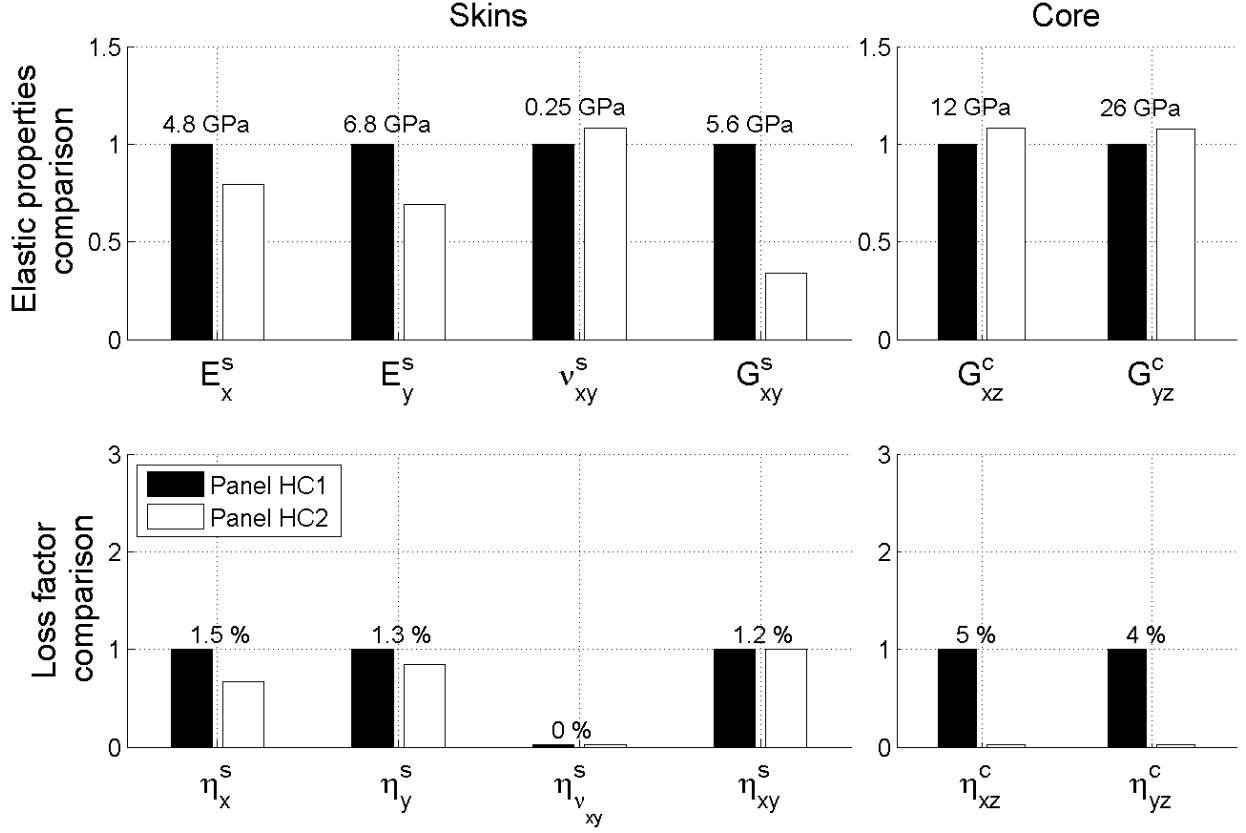


Figure 8: Comparison between the skins and core mechanical parameters estimated from panels HC<sub>1</sub> and HC<sub>2</sub>. Numerical values indicated as references black bars correspond to the results obtained for the panel HC<sub>1</sub>. White bars represent the ratio of the estimated value for HC<sub>2</sub> relatively to the one estimated for HC<sub>1</sub>.

elasticity modulus of steel ( $\approx 210$  GPa [49]). Since panel CC is a metallic sandwich panel, its loss factors are much lower than those of panels HC<sub>1</sub> and HC<sub>2</sub>. The residual differences on eigenfrequencies and dampings are shown in Fig. 11. It can be seen that the uncertainty on the estimation of damping is large  $\left\langle \left| \frac{\Delta \alpha_n}{\alpha_n} \right| \right\rangle = 34\%$  and increases with frequency. Since the estimated structural loss factors are very low, the measured modal dampings are very sensitive to the way the panel is suspended (thin wires) and to damping due to acoustical radiation. Moreover, these damping mechanisms are also mode-dependant. In the high-frequency range, a systematic discrepancy appears between the measured and the (numerically) modelled damping factors. For the equivalent homogeneous plate, the coincidence frequency  $f_c$  is estimated to be approximately 4 kHz. Damping due to acoustical radiation increases as the modal frequency comes close to  $f_c$ . In the same spirit, the difference between  $f_n^{XP}$  and  $f_n^{Num}$  seems to be systematically negative by  $\approx 2\%$ . By order of magnitude, this is consistent with air loading in the low frequency range. The relative differences between experimental and numerical modal frequencies remain small  $\left\langle \left| \frac{\Delta f_n}{f_n} \right| \right\rangle = 2.3\%$ .

Based on the estimated values given in Tab. 6, the modal shapes of the extracted modes can be computed by means of the extended Rayleigh-Ritz procedure of Sec. 3.2. In the  $y$ -direction, panel CC has a maximum of 7 nodal lines in the studied frequency range: its side-length is  $4\lambda_y^{\min}$  and  $\lambda_y^{\min} = 5.7$  cm, containing  $\frac{\lambda_y^{\min}}{s_{\text{cell}}} \approx 55$  cells. In the  $x$ -direction, panel CC has a maximum of 6 nodal lines:  $\lambda_x^{\min} = 5$  cm, containing  $\frac{\lambda_x^{\min}}{s_{\text{cell}}} \approx 50$  cells. This is sufficient for the core to

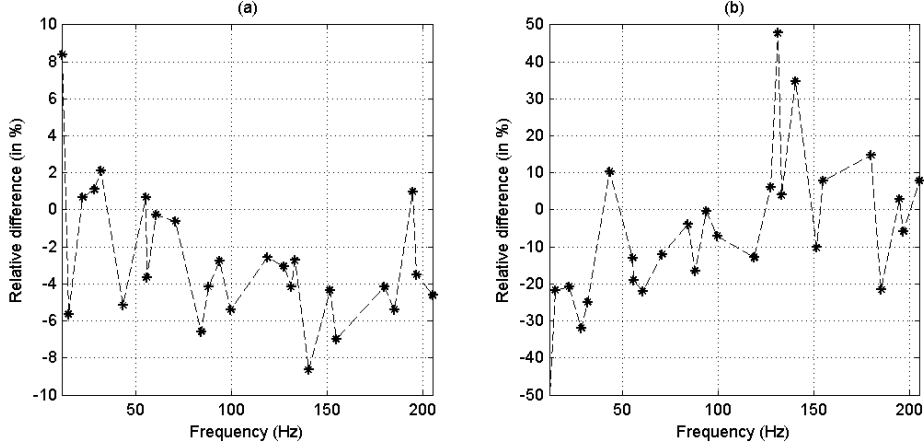


Figure 9: Residual differences on eigenfrequencies (a) and dampings (b) for panel HC<sub>2</sub>. Numerical values have been obtained with elasticity and damping parameters given in Tab. 5.

be considered as homogeneous. Also,  $\frac{2h}{\lambda_{\min}} = 6.6 \times 10^{-2} < 0.08$  in the  $y$ - direction, and  $\frac{2h}{\lambda_{\min}} = 7.52 \times 10^{-2} < 0.08$  in the  $y$ - direction, which is also sufficient for modal frequencies to be predicted by the thick-plate theory. In the frequency range under consideration, the core of panel CC can be considered as homogeneous and the high modal frequencies are expected to be well predicted by the thick-plate theory.

Results of the sensitivity analysis are presented in Fig. 12 for panel CC. It can be seen that sensitivities to the out-of-plane properties are very low compared to sensitivities relative to the in-plane properties and also as compared to sensitivities to the out-of-plane properties of panels HC<sub>1</sub> and HC<sub>2</sub>. This means that the modal frequencies and modal dampings are hardly influenced by the out-of-plane complex moduli. As a consequence, the estimations of these parameters must be interpreted very carefully. In this case the core material is too stiff (the last hypothesis listed in Sec. 2.1 is not valid): the out-of-plane shear moduli are too high to allow for their precise identification.

## 8. Conclusion

In this paper, a method for the measurement of six elasticity moduli and six loss-factors of the constituents of a three-layer symmetrical sandwich material, namely  $E_x^s, E_y^s, \nu_{xy}^s, G_{xy}^s, G_{xz}^c, G_{yz}^c$ , has been presented (sandwich structure in the  $xy$ -plane). The method directly extends the work of de Visscher *et al.* [23] by proposing a means to measure also *out-of-plane* complex moduli. It continues the work of Bastos *et al.* [33] with the inclusion of loss factors in the mixed experimental / numerical procedure. Compared to the work of Pagnacco *et al.* [26] and to that of Matter *et al.* [27, 36], the present method does not require full-field measurements and is thus much simpler to implement and faster to execute. Compared to the method of [29, 37, 66], High Resolution Modal Analysis allows for more modal data to be extracted and used for the estimation of the elastic and damping properties of sandwich materials. Moreover, residuals obtained in the present paper for modal frequencies ( $\approx 1\%$ ) and dampings ( $\approx 10\%$ ) are fully in agreement with residuals obtained in [36–38] which estimate mechanical parameters of sandwich panels using much less modal data than does the present study.

The method is suited to sandwich structures with heterogeneous cores (*e.g.* honeycomb cores) provided that the panels on which the tests are performed satisfy several geometrical requirements. It has been validated successfully on virtual plates. Results obtained on three sandwich panels suggest that the dynamic behaviour of the sandwich material can be accurately modelled using (1) an equivalent homogeneous plate modelled with first order shear deformation theory, (2) a simple hysteretic model of the type “ $E(1 + j\eta)$ ” for each constituent material and (3) the “light damping” hypothesis for the panel. The consistency of the results with those obtained by static measurements, or on two different panels having the same sandwich structure, also contributes to the validation of the method. The extensive sensitivity

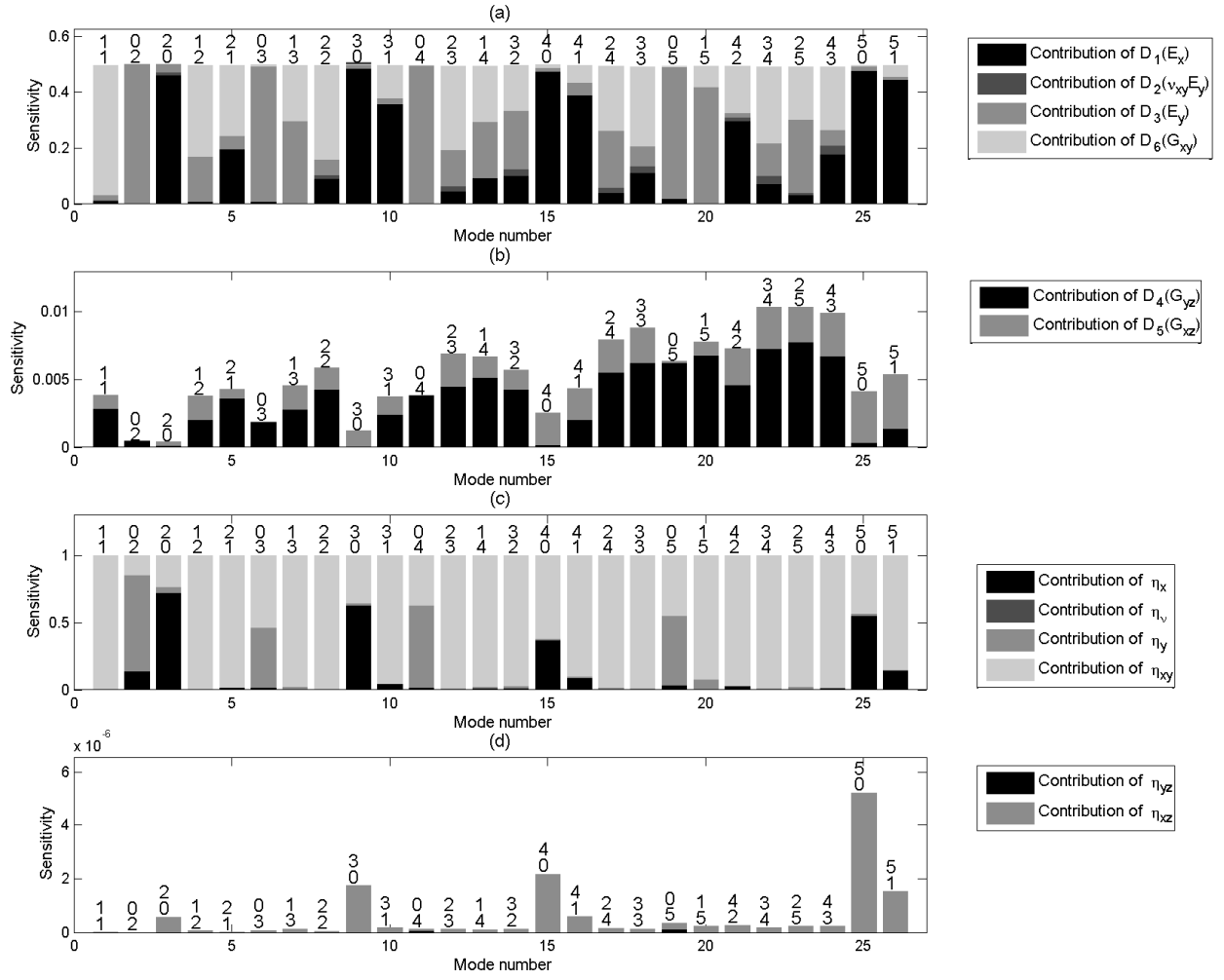


Figure 10: Sensitivities of the modal frequencies to the in-plane (a) and out-of-plane (b) elasticity parameters. Sensitivities of the modal dampings to the in-plane (c) and out-of-plane (d) loss factors for panel HC<sub>2</sub>. Modes are identified by the numbers of their nodal lines in the x- and y-direction (top and bottom numbers respectively, on top of each bar).

analysis combined with the examination of the residual differences left by the optimisation process yields the degree of confidence that can be attributed to the value of each extracted mechanical parameter.

Since this method is simple and needs no heavy experimental apparatus, it is suited to the in-line control of the production of sandwich-materials. This method can also replace classical tensile tests (with some profit) and also, within the frame of the proposed model, the dynamical mechanical analysis (DMA) for the measurement of elastic and damping material properties.

## Acknowledgements

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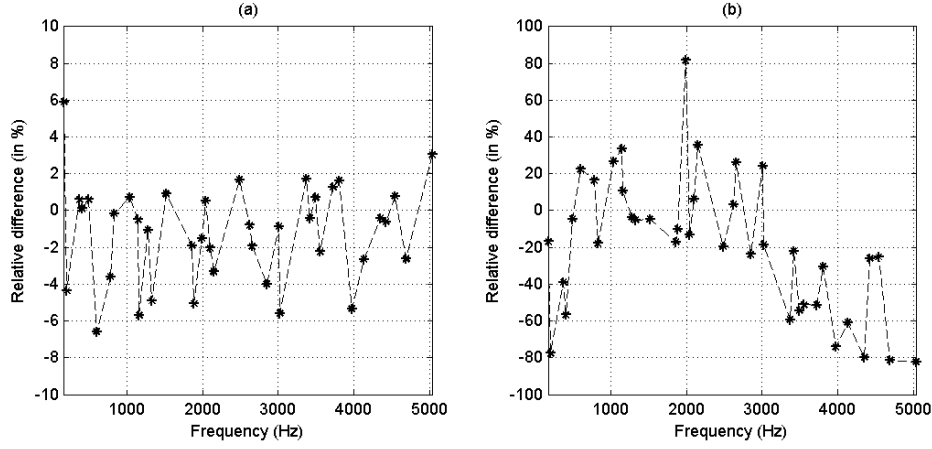


Figure 11: Residual difference on eigenfrequencies (a) and dampings (b) for panel CC. Numerical values have been obtained using elasticity and damping parameters given in Tab. 6.

### A. Generation of the orthonormal polynomial basis

The orthonormal polynomial basis  $\{p_i(\kappa)\}_{i \in [0, N]}$  [55, 56] used in the extended Rayleigh-Ritz procedure of Sec. 3 is generated by an iterative Gram-Schmidt process as follows:

$$\begin{cases} i = 0 & \tilde{p}_0(\kappa) = \frac{1}{\sqrt{2}} \\ i = 1 & \tilde{p}_1(\kappa) = (\kappa - \langle \kappa p_0(\kappa), p_0(\kappa) \rangle) p_0(\kappa) \\ i > 1 & \tilde{p}_i(\kappa) = (\kappa - \langle \kappa p_{i-1}(\kappa), p_{i-1}(\kappa) \rangle) p_{i-1}(\kappa) - \langle \kappa p_{i-2}(\kappa), p_{i-2}(\kappa) \rangle p_{i-2}(\kappa) \end{cases} \quad (25)$$

The scalar product between two polynomials  $a(\kappa)$  and  $b(\kappa)$  is:  $\langle a(\kappa), b(\kappa) \rangle = \int_{-1}^1 a(\kappa)b(\kappa)d\kappa$ . The normalized and not-normalized versions of the  $i^{\text{th}}$  element of the polynomial basis are denoted respectively by  $p_i(\kappa)$  and  $\tilde{p}_i(\kappa)$ . The former is derived from the latter by:  $\tilde{p}_i(\kappa) = \sqrt{\langle \tilde{p}_i(\kappa), \tilde{p}_i(\kappa) \rangle} p_i(\kappa)$ . The basis is orthonormal since the following equation is satisfied:  $\forall (i, j) \in [0, N]^2 \quad \langle p_i(\kappa), p_j(\kappa) \rangle = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker symbol.

### B. Analytical expressions of the derivatives of $T$ and $U$

The matrices  $\mathbf{K}$  and  $\mathbf{M}$  of the eigenvalue problem  $[\mathbf{K} - 4\pi^2 f^2 \mathbf{M}]\mathbf{q} = 0$  (Sec. 3.2) are derived from the analytical expressions of the derivatives of  $T$  and  $U$  relatively to the generalised displacements  $L_{ij}$ ,  $M_{ij}$ ,  $N_{ij}$ . Those are related to the "natural" displacements  $\Phi_x$ ,  $\Phi_y$ , and  $w_0$  by:

$$\Phi_x(x, y) = \sum_{i,j} L_{ij} p_i(x) p_j(y) \quad \Phi_y(x, y) = \sum_{i,j} M_{ij} p_i(x) p_j(y) \quad w_0(x, y) = \sum_{i,j} N_{ij} p_i(x) p_j(y) \quad (26)$$

The derivatives of the kinetic energy  $T = \frac{\rho^H \omega^2}{2} \iint_{(S)} \left[ \frac{h^3}{12} (\Phi_x^2 + \Phi_y^2) + h w_0^2 \right] dx dy$  are:

$$\begin{cases} \frac{\partial T}{\partial L_{ij}} = \frac{\rho^H l_x l_y h \omega^2}{4} L_{ij} \\ \frac{\partial T}{\partial M_{ij}} = \frac{\rho^H l_x l_y h \omega^2}{4} M_{ij} \\ \frac{\partial T}{\partial N_{ij}} = \frac{\rho^H l_x l_y h \omega^2}{4} N_{ij} \end{cases} \quad (27)$$



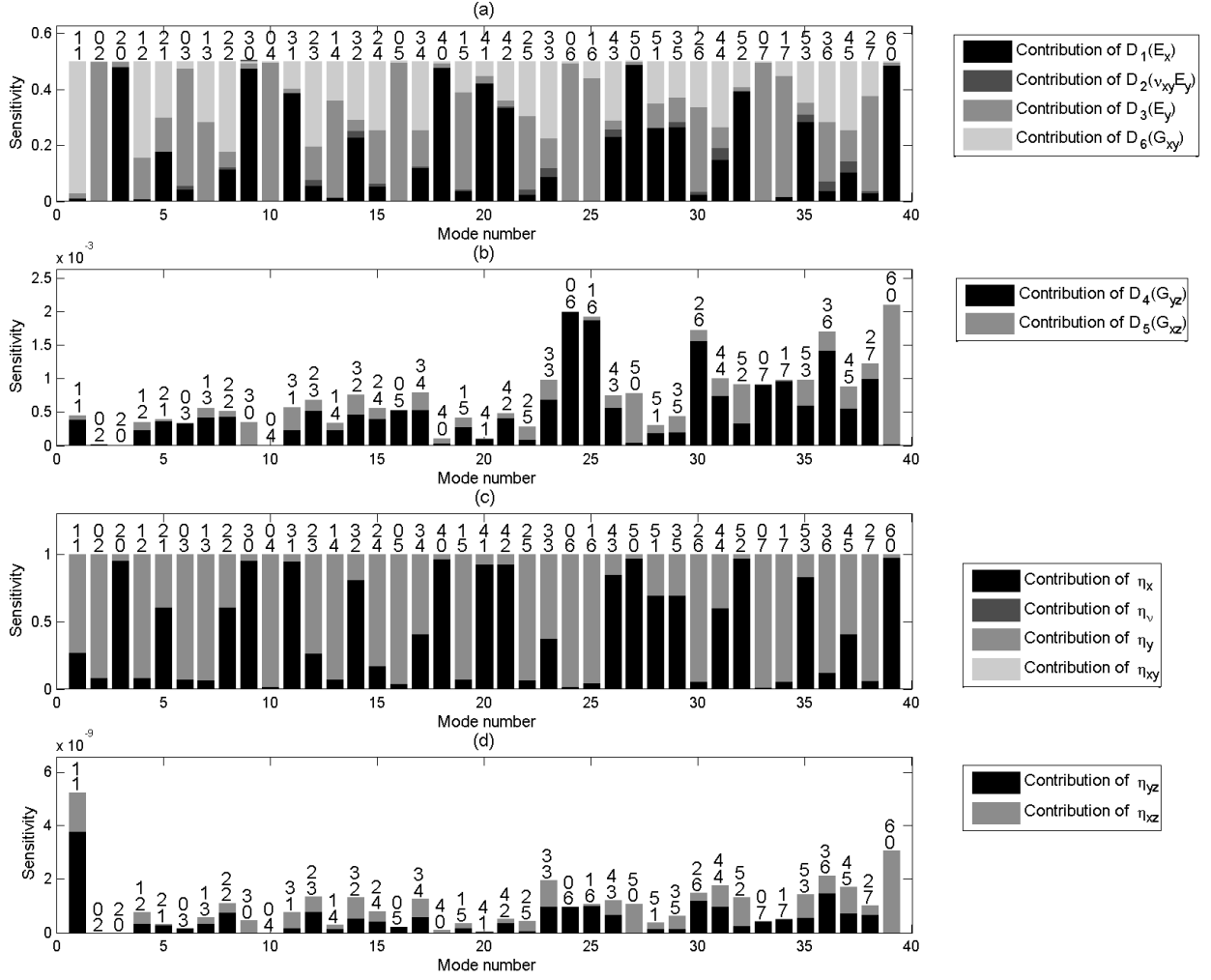


Figure 12: Sensitivities of the modal frequencies to the in-plane (a) and out-of-plane (b) elasticity parameters. Sensitivities of the modal dampings to the in-plane (c) and out-of-plane (d) loss factors for panel CC. Modes are identified by the numbers of their nodal lines in the x- and y-directions (top and bottom numbers respectively, on top of each bar).

The simplicity of the formulas is due to the fact that no space derivatives appear in the expression of the kinetic energy and that all products  $p_i p_j$  ( $i \neq j$ ) cancel out once integrated (orthogonality of the polynomials).

The derivatives of the potential energy  $U$  (given by Eq. 7) are:

$$\left\{ \begin{array}{l} \frac{\partial U}{\partial L_{ij}} = \sum_{k=0}^N \left[ -\frac{l_x}{2} D_4 N_{ik} I(j, k) + \frac{l_x}{l_y} D_4 L_{ik} J(j, k) - \frac{l_y}{2} D_5 M_{kj} I(i, k) + \frac{l_y}{l_x} D_5 L_{kj} J(i, k) \right] \\ \frac{\partial U}{\partial M_{ij}} = \frac{l_x l_y}{4} D_5 M_{ij} + \sum_{k=0}^N \left[ \frac{l_y}{l_x} D_1 M_{kj} J(i, k) - \frac{l_y}{2} D_5 L_{kj} I(k, i) + \frac{l_x}{l_y} D_6 M_{ik} J(j, k) \right] + \dots \\ \dots \sum_{k,l} \left[ \frac{D_2}{2} I(i, k) I(l, j) + D_6 I(k, i) I(j, l) \right] N_{kl} \\ \frac{\partial U}{\partial N_{ij}} = \frac{l_x l_y}{4} D_4 N_{ij} + \sum_{k=0}^N \left[ \frac{l_x}{l_y} D_3 N_{ik} J(k, j) - \frac{l_x}{2} D_4 L_{ik} I(k, j) + \frac{l_y}{l_x} D_6 N_{kj} J(k, i) \right] + \dots \\ \dots \sum_{k,l} \left[ \frac{D_2}{2} I(k, i) I(j, l) + D_6 I(i, k) I(l, j) \right] M_{kl} \end{array} \right. \quad (28)$$

where  $\sum_{k,l}$  stands for  $\sum_{(k,l) \in [0,N]^2}$  and where the following integrals have been introduced:

$$I(i, j) = \langle \frac{dp_i(\kappa)}{d\kappa}, p_j(\kappa) \rangle = \int_{-1}^1 \frac{dp_i(\kappa)}{d\kappa} p_j(\kappa) d\kappa \quad , \quad J(i, j) = \langle \frac{dp_i(\kappa)}{d\kappa}, \frac{dp_j(\kappa)}{d\kappa} \rangle = \int_{-1}^1 \frac{dp_i(\kappa)}{d\kappa} \frac{dp_j(\kappa)}{d\kappa} d\kappa \quad (29)$$

### C. Analytical expressions of $t_n$ and $u_k^n$

The explicit expression of the coefficients  $t_n$  and  $u_k^n$ , representing respectively the  $n$ -th modal mass (for some normalised displacement) and the  $k$ -contribution to the  $n$ -th modal stiffness, are necessary to compute the numerical modal dampings and the sensitivities of the modal frequencies and dampings to the rigidities and loss factors respectively, as explained in Secs. 3.3 and 5.4. The coordinates of the  $n^{\text{th}}$  modal shape are denoted by  $\{L_{lm}^n, M_{lm}^n, N_{lm}^n\}$ . The calculation is sketched in section 3.3. The expression of  $t_n$  is:

$$t_n = \frac{\rho^H l_x l_y h}{8} \sum_{l,m} \left[ (L_{lm}^n)^2 + \frac{h^2}{12} ((M_{lm}^n)^2 + (N_{lm}^n)^2) \right] \quad (30)$$

The expressions of  $\{u_k^n\}_{k \in [1,6]}$  are:

$$\left\{ \begin{array}{l} u_1^n = \frac{l_y}{2l_x} \sum_{l,m} M_{lm}^n \left[ \sum_{p=0}^N M_{pm}^n J(l, p) \right] \\ u_2^n = \frac{1}{2} \sum_{l,m} M_{lm}^n \left[ \sum_{p,q} N_{pq}^n I(l, p) I(l, q) \right] \\ u_3^n = \frac{l_x}{2l_y} \sum_{l,m} N_{ml}^n \left[ \sum_{p=0}^N N_{lp}^n J(m, p) \right] \\ u_4^n = \frac{l_x l_y}{8} \sum_{l,m} (N_{ml}^n)^2 - \frac{l_x}{2} \sum_{l,m} N_{ml}^n \left[ \sum_{p=0}^N L_{lp}^n I(p, m) \right] + \frac{l_x}{2l_y} \sum_{l,m} L_{ml}^n \left[ \sum_{p=0}^N L_{lp}^n J(m, p) \right] \\ u_5^n = \frac{l_x l_y}{8} \sum_{l,m} (M_{ml}^n)^2 - \frac{l_y}{2} \sum_{l,m} M_{ml}^n \left[ \sum_{p=0}^N L_{pm}^n I(p, l) \right] + \frac{l_y}{2l_x} \sum_{l,m} L_{ml}^n \left[ \sum_{p=0}^N L_{pm}^n J(l, p) \right] \\ u_6^n = \frac{l_x}{2l_y} \sum_{l,m} M_{ml}^n \left[ \sum_{p=0}^N M_{lp}^n J(m, p) \right] + \sum_{l,m} M_{ml}^n \left[ \sum_{p,q} N_{pq}^n I(p, l) I(m, q) \right] + \frac{l_y}{2l_x} \sum_{l,m} N_{ml}^n \left[ \sum_{p=0}^N N_{pm}^n J(l, p) \right] \end{array} \right. \quad (31)$$

where the  $I(l, p)$  and  $J(l, p)$  are defined at the end of B.

#### D. Nomenclature

$l_x$	(m)	Length of the $x$ -side of the plate
$l_y$	(m)	Length of the $y$ -side of the plate
$h^s$	(m)	Skin thickness
$h^c$	(m)	Core thickness
$h = h^c + 2h^s$	(m)	Sandwich panel thickness
$\rho^s$	(kg×m <sup>-3</sup> )	Skin mass density
$\rho^c$	(kg×m <sup>-3</sup> )	Core mass density
$\rho^H = 1/h \times (h^c\rho^c + 2h^s\rho^s)$	(kg×m <sup>-3</sup> )	Equivalent homogeneous plate mass density
$s_{\text{cell}}$	(m)	Characteristic side-length of the core-cells

Table 7: Notations for the geometrical and mass parameters of the panels

	Core	Skin	Equivalent homogeneous plate
Young modulus in the $x$ -direction	$\underline{E}_x^c = E_x^c(1 + j\eta_x^c)$	$\underline{E}_x^s = E_x^s(1 + j\eta_x^s)$	$\underline{E}_x^H = E_x^H(1 + j\eta_x^H)$
Young modulus in the $y$ -direction	$\underline{E}_y^c = E_y^c(1 + j\eta_y^c)$	$\underline{E}_y^s = E_y^s(1 + j\eta_y^s)$	$\underline{E}_y^H = E_y^H(1 + j\eta_y^H)$
Young modulus in the $z$ -direction	$\underline{E}_z^c = E_z^c(1 + j\eta_z^c)$	$\underline{E}_z^s = E_z^s(1 + j\eta_z^s)$	$\underline{E}_z^H = E_z^H(1 + j\eta_z^H)$
Shear modulus in the $xy$ -plane	$\underline{G}_{xy}^c = G_{xy}^c(1 + j\eta_{xy}^c)$	$\underline{G}_{xy}^s = G_{xy}^s(1 + j\eta_{xy}^s)$	$\underline{G}_{xy}^H = G_{xy}^H(1 + j\eta_{xy}^H)$
Poisson ratio in the $xy$ -plane	$\underline{\nu}_{xy}^c = \nu_{xy}^c(1 + j\eta_{\nu_{xy}}^c)$	$\underline{\nu}_{xy}^s = \nu_{xy}^s(1 + j\eta_{\nu_{xy}}^s)$	$\underline{\nu}_{xy}^H = \nu_{xy}^H(1 + j\eta_{\nu_{xy}}^H)$
Shear modulus in the $xz$ -plane	$\underline{G}_{xz}^c = G_{xz}^c(1 + j\eta_{xz}^c)$	$\underline{G}_{xz}^s = G_{xz}^s(1 + j\eta_{xz}^s)$	$\underline{G}_{xz}^H = G_{xz}^H(1 + j\eta_{xz}^H)$
Poisson ratio in the $xz$ -plane	$\underline{\nu}_{xz}^c = \nu_{xz}^c(1 + j\eta_{\nu_{xz}}^c)$	$\underline{\nu}_{xz}^s = \nu_{xz}^s(1 + j\eta_{\nu_{xz}}^s)$	$\underline{\nu}_{xz}^H = \nu_{xz}^H(1 + j\eta_{\nu_{xz}}^H)$
Shear modulus in the $yz$ -plane	$\underline{G}_{yz}^c = G_{yz}^c(1 + j\eta_{yz}^c)$	$\underline{G}_{yz}^s = G_{yz}^s(1 + j\eta_{yz}^s)$	$\underline{G}_{yz}^H = G_{yz}^H(1 + j\eta_{yz}^H)$
Poisson ratio in the $yz$ -plane	$\underline{\nu}_{yz}^c = \nu_{yz}^c(1 + j\eta_{\nu_{yz}}^c)$	$\underline{\nu}_{yz}^s = \nu_{yz}^s(1 + j\eta_{\nu_{yz}}^s)$	$\underline{\nu}_{yz}^H = \nu_{yz}^H(1 + j\eta_{\nu_{yz}}^H)$

Table 8: Notations for the complex mechanical parameters (elastic parameters and loss factors) of the panels ( $j^2 = -1$ )

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$u(x, y, z)$	(m)	Displacement in the $x$ -direction
$v(x, y, z)$	(m)	Displacement in the $y$ -direction
$w(x, y, z)$	(m)	Displacement in the $z$ -direction
$\Phi_x(x, y)$	(rad)	Rotation around the $x$ -axis (Reissner-Mindlin hypothesis)
$\Phi_y(x, y)$	(rad)	Rotation around the $y$ -axis (Reissner-Mindlin hypothesis)
$w_0(x, y)$	(m)	Displacement in the $z$ -direction (Reissner-Mindlin hypothesis)
$U$	(J)	Potential energy
$\Delta U$	(J)	Energy lost per cycle
$T$	(J)	Kinetic energy
$D_1 = \frac{E_x^H h^3}{12(1 - \nu_{xy}\nu_{yx})}$	(N×m)	Plate rigidity in the $x$ -direction
$D_2 = \frac{\nu_{xy} E_y^H h^3}{6(1 - \nu_{xy}\nu_{yx})}$	(N×m)	Plate rigidity in the $xy$ -plane
$D_3 = \frac{E_y^H h^3}{12(1 - \nu_{xy}\nu_{yx})}$	(N×m)	Plate rigidity in the $y$ -direction
$D_4 = 2\kappa_{yz}^2 h G_{yz}^H$	(N×m)	Plate rigidity in the $yz$ -plane
$D_5 = 2\kappa_{xz}^2 h G_{xz}^H$	(N×m)	Plate rigidity in the $xz$ -plane
$D_6 = \frac{G_{xy}^H h^3}{6}$	(N×m)	Plate rigidity in the $xy$ -plane
$\kappa_{xz}$	-	Shear correction factor in the $xz$ -plane
$\kappa_{yz}$	-	Shear correction factor in the $yz$ -plane
$\eta_v^{c,s,H} = \eta_{\nu_{xy}}^{c,s,H} + \eta_y^{c,s,H} = \eta_{\nu_{yx}}^{c,s,H} + \eta_x^{c,s,H}$	-	Global loss factor due to the Poisson ratio effects in the $xy$ -plane
$\eta_1 = \eta_x^H$	-	Loss factor in the $x$ -direction
$\eta_2 = \eta_v^H$	-	Global loss factor due to the Poisson ratio effects in the $xy$ -plane
$\eta_3 = \eta_y^H$	-	Loss factor in the $y$ -direction
$\eta_4 = \eta_{yz}^H$	-	Loss factor in the $yz$ -plane
$\eta_5 = \eta_{xz}^H$	-	Loss factor in the $xz$ -plane
$\eta_6 = \eta_{xy}^H$	-	Loss factor in the $xy$ -plane
$f$	(Hz)	Frequency
$\omega = 2\pi f$	(rad×s <sup>-1</sup> )	Angular frequency
$\lambda$	(m)	Wavelength of the flexural vibrations
$\lambda_x^{\min}$	(m)	Minimal wavelength of the flexural vibrations in the $x$ -direction in the frequency range under study
$\lambda_y^{\min}$	(m)	Minimal wavelength of the flexural vibrations in the $y$ -direction in the frequency range under study

Table 9: Notations used in the dynamical model of the panels

$\mathbf{K}$	(N×m <sup>-1</sup> )	Stiffness matrix
$\mathbf{M}$	(kg)	Mass matrix
$\mathbf{C}$	(N×m <sup>-1</sup> ×s <sup>-1</sup> )	Damping matrix
$f_n^{\text{XP}}$	(Hz)	Experimentally obtained modal frequency of the $n^{\text{th}}$ -mode
$\alpha_n^{\text{XP}}$	(s <sup>-1</sup> )	Experimentally obtained modal damping of the $n^{\text{th}}$ -mode
$f_n^{\text{Num}}$	(Hz)	Numerically obtained modal frequency of the $n^{\text{th}}$ -mode
$\alpha_n^{\text{Num}}$	(s <sup>-1</sup> )	Numerically obtained modal damping of the $n^{\text{th}}$ -mode
$\xi_n^{\text{Num}}$	(m)	Numerically obtained modal shape of the $n^{\text{th}}$ -mode
$\mu$	-	Modal overlap
$\mathcal{P}_C$	-	Conservative system associated to the plate
$U_n^C$	(J)	Potential energy of the $n^{\text{th}}$ -mode of $\mathcal{P}_C$
$T_n^C$	(J)	Kinetic energy of the $n^{\text{th}}$ -mode of $\mathcal{P}_C$
$\mathcal{P}_{\text{NC}}$	-	Non-conservative system associated to the plate
$U_n^{\text{NC}}$	(J)	Potential energy of the $n^{\text{th}}$ -mode of $\mathcal{P}_{\text{NC}}$
$T_n^{\text{NC}}$	(J)	Kinetic energy of the $n^{\text{th}}$ -mode of $\mathcal{P}_{\text{NC}}$
$\Delta U_n^{\text{NC}}$	(J)	Energy lost per cycle by the $n^{\text{th}}$ -mode of $\mathcal{P}_{\text{NC}}$
$Q$	-	Order of the polynomial basis (Rayleigh-Ritz procedure)
$p_i(x)$	-	$i^{\text{th}}$ element of the polynomial basis in the $x$ -direction (Rayleigh-Ritz procedure)
$p_j(y)$	-	$j^{\text{th}}$ element of the polynomial basis in the $y$ -direction (Rayleigh-Ritz procedure)
$L_{ij}$	(rad)	Coordinates of $\Phi_x(x, y)$ in the polynomial basis (Rayleigh-Ritz procedure)
$M_{ij}$	(rad)	Coordinates of $\Phi_y(x, y)$ in the polynomial basis (Rayleigh-Ritz procedure)
$N_{ij}$	(m)	Coordinates of $w_0(x, y)$ in the polynomial basis (Rayleigh-Ritz procedure)

Table 10: Notations used in the numerical model of the panels